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Géométrie de la projectivisation des idéaux et applications aux problèmes de birationalité

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Al andar se hace camino
y al volver la vista atrás
se ve la senda que nunca
se ha de volver a pisar.

Antonio Machado, *Caminante no hay camino*

En marchant se fait le chemin
et c'est en se retournant
que l'on peut contempler le sentier
que l'on n'aura jamais plus l'occasion d'emprunter.

Traduction personnelle

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Résumé grand public

L'explosion récente des capacités de calculs numériques qui s'est produite au cours des trente dernières années a renversé les perspectives de recherches et développements. La simulation numérique permet désormais une estimation plus fiable et beaucoup plus rapide de résultats autrefois inatteignables. Dans cette thèse, nous appliquons ce principe très général à un domaine des mathématiques, la géométrie algébrique, qui concerne l'étude des équations polynomiales. Cela nous permet de prévoir l'existence de configurations géométriques inattendues apportant ainsi un point de vue original sur des sujets classiques de géométrie. En plus de ces simulations, nous nous attachons à démontrer mathématiquement ces phénomènes annoncés numériquement. Nos résultats concernent aussi le développement de méthodes numériques visant à améliorer davantage les capacités et la rapidité des simulations dans ce domaine.

Large audience abstract

The recent growth in capacity of numerical calculus over the last thirty years has brought a major change in perspective regarding matters of research and development. Numerical simulation allows a more reliable and faster estimation of results unattainable beforehand. In this thesis, we apply this very general principle to a domain of mathematics, algebraic geometry which concerns the study of polynomial equations. This allows us to predict unexpected geometrical configurations bringing an original point of view to classical geometrical subjects. In addition of those simulations, we focus on proving mathematically those predicted phenomenons. Our results also concern the development of numerical methods aiming to further improve the capacity and the rapidity of simulations in this area.

Résumé

Dans cette thèse, nous interprétons géométriquement la torsion de l'algèbre symétrique d'un faisceau d'idéaux $\mathcal{I}_Z$ d'un schéma Z défini par $n+1$ équations dans une variété n-dimensionnelle. Ceci revient à étudier la géométrie de la projectivisation de $\mathcal{I}_Z$. Les applications de ce point de vue concernent en particulier le domaine des transformations birationnelles de l'espace projectif de dimension 3 au sujet duquel nous construisons des transformations birationnelles explicites qui ont le même degré algébrique que leur inverse, le domaine des courbes libres et presque-libres au sujet duquel nous généralisons une caractérisation des courbes libres en étendant les notions de nombre de Milnor et de nombre de Tjurina. Nous abordons aussi le sujet des hypersurfaces homaloides, notre motivation initiale, au sujet duquel nous exhibons en particulier une courbe homaloide de degré 5 en caractéristique 3. La dernière application concerne le calcul de l'inverse d'une transformation birationnelle.

Mots clés: Géométrie algébrique, Algèbre commutative, Théorie des singularités, Transformations birationelles, Hypersurfaces homaloïdes, courbes libres et presque libres, algèbre de Rees et algèbre symmétrique, Syzygies, Résolutions

Title of the thesis : Geometry of the projectivization of ideals and applications to problems of birationality

Abstract

In this thesis, we interpret geometrically the torsion of the symmetric algebra of the ideal sheaf $\mathcal{I}_Z$ of a scheme Z defined by $n+1$ equations in an n-dimensional variety. This is equivalent to study the geometry of the projectivization of $\mathcal{I}_Z$. The applications of this point of view concern, in particular, the topic of birational maps of the projective space of dimension 3 for which we construct explicit birational maps that have the same algebraic degree as their inverse, free and nearly-free curves for which we generalise a characterization of free curves by extending the notion of Milnor and Tjurina numbers. We tackle also the topic of homaloidal hypersurfaces, our original motivation, for which we produce in particular a homaloidal curve of degree 5 in characteristic 3. The last application concerns the computation of the inverse of a birational map.

Keywords: Algebraic Geometry, Commutative algebra, Singularity theory, Birational maps, Homaloidal hypersurfaces, free and nearly free curves, Symmetric and Rees algebra, Syzygies, Resolutions

Contents

Introduction

We introduce first the basic objects and topics we deal with and we present our contribution to these topics in the second section. Our objects of interest are rational maps between algebraic varieties over a field k. A rational map $\Phi : X \dashrightarrow Y$ is an equivalence class of pairs $\langle U, \Phi \rangle$ where U is a nonempty open subset of X, Φ_U is a morphism of U to Y, and where $\langle U, \Phi_U \rangle$ and $\langle V, \Phi_V \rangle$ are equivalent if Φ_U and Φ_V agree on $U \cap V$. In general, Φ (more precisely a representative of Φ) is not defined everywhere on X and the closed subset Z of X where Φ is not defined is called the base locus of Φ.

Letting $U = X \backslash Z$ so that the representative $\Phi_U : U \to Y$ of Φ is a morphism, the closure of the image of Φ_U in Y does not depend on the representative of Φ and is called the image of Φ, denoted by $\mathrm{Im}(\Phi)$. Given also a closed subvariety W of Y, the closure in X of the inverse image $\Phi_U^{-1}(W)$, denoted by $\Phi^{-1}(W)$ is called the inverse transform of W under Φ.

The graph Γ_Φ of Φ is defined as the closure

$$\overline{\{\big(x, \Phi(x)\big),\ x \in U\}} \subset X \times Y$$

of the graph of the morphism $\Phi_U : U \to Y$ in $X \times Y$ (it does not depend on the representative of Φ as well).

Since dominant rational maps $\Phi : X \dashrightarrow Y$ and $\Psi : Y \dashrightarrow X$ can be composed, we are especially interested in the case when this composition has the identity map as representative. In this case, we say that Φ (and Ψ) is birational.

Assuming that the varieties X and Y are irreducible of the same dimension and that $\Phi : X \dashrightarrow Y$ is dominant (i.e. $\mathrm{Im}(\Phi)$ is dense in Y), we define the topological degree of Φ as the degree

$$d_t(\Phi) = [\mathrm{k}(X) : \mathrm{k}(Y)]$$

of the field extension of $\mathrm{k}(X)$ over $\mathrm{k}(Y)$ the respective fraction fields of X and Y (cf Proposition 1.3.8 for a justification that the integer we just defined is indeed the actual topological degree of a morphism $\Phi : U \to Y$). In this perspective, the property of Φ to be birational is equivalent to $d_t(\Phi) = 1$.

Our main source of examples and applications comes from the situation where $X = Y = \mathbb{P}^n$. In this case, a preferred representative of a rational map is defined by $n+1$ homogeneous polynomials $\phi_0, \ldots, \phi_n \in \mathrm{k}[x_0, \cdots, x_n]$ of the same degree δ without common component.

Example 1. Let $\tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the map defined by the polynomials x_1x_2, x_0x_2 and x_0x_1. The base locus Z of Φ is the union of the three points $\mathbb{V}(x_1, x_2) \cup$

$\mathbb{V}(x_0, x_2) \cup \mathbb{V}(x_0, x_1)$ (in the following, the notation $\mathbb{V}(s)$ for s a polynomial or more generally a section of a sheaf always stands for the zero locus of s in the ambient variety).

Moreover $\tau \circ \tau : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ is defined by the polynomials

$$(x_0x_1x_2)x_0, (x_0x_1x_2)x_1, (x_0x_1x_2)x_2.$$

But considering the map id defined by the polynomials x_0, x_1, x_2, we see that $\tau \circ \tau$ and id coincide over $\mathbb{P}^2 \backslash \big(\mathbb{V}(x_0) \cup \mathbb{V}(x_1) \cup \mathbb{V}(x_2)\big)$. So, since rational maps are equivalence classes, a preferred representative of $\tau \circ \tau$ is the identity map defined by x_0, x_1, x_2.

Actually, the map τ is quite special, at least because it is an involution (i.e. $\tau = \tau^{-1}$). It is called the *standard Cremona map*.

More generally, X (respectively Y) being a smooth subvariety of $\mathbb{P}^n$, (respectively a smooth subvariety of $\mathbb{P}^m$) we are interested in the multidegree of Φ. Roughly letting $k = \dim(X)$ and $i \in \{0, \dots, k\}$, we can define a number $d_i(\Phi)$, called i^{th} projective degree of Φ, as follows

$$d_i(\Phi) = \operatorname{card}\big(H_1^i \cap \Phi^{-1}(H_2^{k-i})\big)$$

where H_1^i is a general $(n-i)$-plane of $\mathbb{P}^n$ and H_2^{k-i} is a general $(m-k+1)$-plane of $\mathbb{P}^m$. Less roughly, one has to use intersection theory, cf. Definition 1.3.12. The sequence $\big(d_k(\Phi), \dots, d_0(\Phi)\big)$ is called the multidegree of Φ. When there is no ambiguity about the rational map Φ, we simply denote by $(d_k, \dots, d_0)$ the multidegree of Φ.

Example 2. When $\Phi : \mathbb{P}_1^n \dashrightarrow \mathbb{P}_2^n$ is defined by $n+1$ homogeneous polynomials $\phi_0, \dots, \phi_n \in \mathrm{k}[x_0, \cdots, x_n]$ of the same degree δ without common factor, the first projective degree $d_1(\Phi)$ of Φ is the cardinality of the intersection of a hypersurface $\mathbb{V}(\sum_{i=0}^{n} a_i\phi_i)$ of degree δ with $n-1$ general hyperplanes. Hence, by Bézout's theorem, $d_1(\Phi) = \delta$ is the algebraic degree of the map Φ, i.e. the degree of each polynomials defining Φ.

Example 3. Let $\Phi : \mathbb{P}_1^3 \dashrightarrow \mathbb{P}_2^3$ be a rational map of multidegree $(1, d_2, d_1, 1)$ for a given integer n. In particular Φ is birational of inverse Φ^{-1}. But since $\Gamma_\Phi = \Gamma_{\Phi^{-1}}$, we see that the 2nd projective degree $d_2(\Phi)$ of Φ, is also the 1st projective degree $d_1(\Phi^{-1})$ of Φ^{-1}, hence the algebraic degree of Φ^{-1}. In greater generality, given an integer $n \geq 2$ and a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of multidegree $(1, d_{n-1}, \dots, d_1, 1)$, d_{n-1} is the algebraic degree of Φ^{-1}.

Since the composition $\Phi_1 \circ \Phi_2$ of two birational maps is birational, the set of birational maps has group structure whose group law is the composition of maps. This group is called the Cremona group of $\mathbb{P}^n$, denoted by $\mathrm{Bir}(\mathbb{P}^n)$ and its elements (birational maps) are also called Cremona maps. Let us present a problem related to the multidegree of the graph of a Cremona map Φ following [Dol11, 7.1.3]. Let $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a Cremona map and let $(1, d_{n-1}, \dots, d_1, 1)$ be its multidegree.

Theorem 4. *(L.Cremona, Cremona's inequalities) For any $n \geq i, j \geq 0$,*

$$1 \leq d_{i+j} \leq d_i d_j,$$
$$d_{n-i-j} \leq d_{n-i} d_{n-j}.$$

See [Dol11, 7.1.7] for the proof of this result.

Remark 5. [Dol11, 7.1.8] There are more conditions on the multidegree which follow from the irreducibility of the graph Γ of Φ. For example, if k is a field of characteristic 0, by using the *Hodge type inequalities* we get the inequalities

$$d_i^2 \geq d_{i-1} d_{i+1}.$$

These conditions on the multidegree lead to the following problem [SJ68]

Problem A. *Let $(1, d_{n-1}, \ldots, d_1, 1)$ be a sequence of integers satisfying the Cremona inequalities and the Hodge type inequalities. Does there exist a Cremona map with this sequence as multidegree?*

As we will see, our work fits in this question because we construct Cremona maps with given multidegree. The basis of these constructions is to study rational maps whose defining polynomials are the maximal minors of a matrix. Let us explain in more details this situation.

Determinantal rational maps

Let $n \geq 1$ and M be a matrix of size $(n+1) \times n$ whose entries are homogeneous polynomials in $\mathrm{k}[x_0, \cdots, x_n]$ and such that all the entries of each column have the same algebraic degree. Then the $n \times n$-minors of M define a rational map of $\mathbb{P}^n$. These maps are called determinantal rational maps and, when birational, they are called determinantal Cremona maps.

We are especially interested in determinantal Cremona maps under the influence of the two articles [Pan99] and [DH17]. In [Pan99], the author describes in particular families of determinantal Cremona maps such as the families of *determinantal cubo-cubic* of $\mathbb{P}^3$. A Cremona map of this family is defined by the 3×3-minors of a 4×3 matrix with linear entries in $\mathrm{k}[x_0, \ldots, x_3]$ (plus additional conditions) and it has multidegree $(1, 3, 3, 1)$ from where the name *determinantal cubo-cubic* comes from. In [DH17], the two authors describes a family of *determinantal quarto-quartic*. A Cremona map of this family is defined by the 3×3-minors of a 4×3 matrix with two columns of linear entries and one columns of quadratic entries and it has multidegree $(1, 4, 4, 1)$. We were particularly interested in constructing other examples of determinantal maps, since, as we will see, such a study is well accessible by the objects we consider.

Homaloidal hypersurfaces and free divisors

We present now another type of Cremona map of $\mathbb{P}^n$, those whose polynomials are the partial derivatives of a homogeneous polynomial $f \in \mathrm{k}[x_0, \cdots, x_n]$. Even if our results and research embrace eventually a more general point of view, namely, studying the presentation of the base ideal of any rational map, we emphasize that

tackling the domain of homaloidal hypersurfaces was our first motivation. Here the field k is an algebraically closed field of any characteristic and we let $n \geq 1$ be an integer.

Definition 6. Let $f \in \mathrm{k}[x_0, \ldots, x_n]$ be a homogeneous polynomial and for all $i \in \{0, \ldots, n\}$ we let $f_i = \frac{\partial f}{\partial x_i}$ be the partial derivative of f with respect to the ith variables. We call the rational map

$$\begin{aligned} \Phi_f : \quad & \mathbb{P}^n \dashrightarrow \mathbb{P}^n \\ & x \dashrightarrow \big(f_0(x) : \ldots : f_n(x)\big) \end{aligned}$$

the *polar map* of the hypersurface $F = \mathbb{V}(f)$ defined by f in $\mathbb{P}^n$. The topological degree of Φ_f, i.e. its nth projective degree, is called the *polar degree of* F and the ideal of the base locus of Φ_f is called the *jacobian ideal* of F.

The hypersurface F or the polynomial f is called *homaloidal* if Φ_f is birational.

We emphasize that if f has degree d, the partial derivatives f_i have degree $d-1$ (or are 0). However, the algebraic degree of Φ_f may not be equal to $d-1$ since we have to remove common factors of the f_i in order that a representative of Φ_f has a codimension at least 2 base locus. This is the case in particular when f is not *square free*, that is when one of the exponents α_i in the decomposition $f = q_1^{\alpha_1} \ldots q_m^{\alpha_m}$ into irreducible homogeneous polynomials $q_i \in \mathrm{k}[x_0, \ldots, x_n]$ is stricly greater than 1. Stating that f is square free is equivalent to set that the hypersurface $F = \mathbb{V}(f)$ is reduced.

As examples of homaloidal hypersurfaces, we have the foundational result of I.V.Dolgachev classifying the reduced homaloidal curves assuming the base field k is the field of complex numbers $\mathbb{C}$.

Theorem 7. *[Dol00, Theorem 4] The only complex reduced homaloidal curves are the smooth conics, the unions of three general lines and the unions of a smooth conic with one of its tangent.*

Without giving the complete proof of this result, let us explain it in the following paragraphs. Given a polar map $\Phi_f = (f_0 : \ldots : f_n)$, the base locus $Z = \mathbb{V}(f_0, \ldots, f_n)$ of Φ is precisely the singular locus of F, that is, the locus where the tangent space of F has bigger dimension than expected. When we consider curves in $\mathbb{P}^2$, this singular locus is necessarily 0-dimensional. Generalising to a hypersurface having only 0-dimensional singularities, one number permits to classify those singularities, the *Milnor number*:

Definition 8. [Mil68] Let Z be the 0-dimensional singular locus of a hypersurface $F = \mathbb{V}(f)$ of $\mathbb{P}^n$ and let $z \in Z$. Via a change of coordinates, suppose that $z = (1 : 0 : \ldots : 0)$. Set $g_\flat \in \mathrm{k}[x_1, \ldots, x_n]$, the usual deshomogeneisation of a homogeneous polynomial $g \in \mathrm{k}[x_0, \cdots, x_n]$ in the chart $\{x_0 \neq 0\}$.

The *local Milnor number* at z, denoted by $\mu_f(Z, z)$, is defined as

$$\mu_f(Z, z) = \text{length}\left(\mathcal{O}_{\mathrm{k}^n, z} / ((f_\flat)_1, \ldots, (f_\flat)_n)\right) \quad \text{where } (f_\flat)_i = \frac{\partial f_\flat}{\partial x_i}.$$

The *global Milnor number* of F, denoted by $\mu_f(Z)$ is the sum $\sum \mu_f(Z,z)$ over all $z \in Z$.

Example 9. Let $f = x_0(x_1^2 + x_0x_2) \in \mathrm{k}[x_0, x_1, x_2]$ and $F = \mathbb{V}(f) \subset \mathbb{P}^n$ be the union of a smooth conic with one of its tangent. The singular locus Z is equal to $\mathbb{V}(x_1^2 + 2x_0x_2, 2x_0x_1, x_0^2)$ so F is singular at the point $z = (0:0:1)$. Hence, on the chart $\{x_2 \neq 0\}$, we have to compute the length of the module $\mathrm{k}[x_0, x_1]/(x_1^2+2x_0, 2x_0x_1)$. As a module, this quotient is only generated by $1, x_0, x_1$ since, for example $x_0^2 \equiv -\frac{1}{2}x_1^2x_0 \equiv 0 \bmod (x_1^2+2x_0, 2x_0x_1)$. Hence $\mu_f = \mu_f(Z,z) = 3$.

When $\mathrm{k} = \mathbb{C}$, assuming that this singular locus is finite, the following relation is established by A.Dimca and S.Papadima [DP03]:

Theorem 10. *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a square free homogeneous polynomial of degree d and let Φ_f be the polar map of $F = \mathbb{V}(f) \subset \mathbb{P}^n$. Assuming that F has finite base locus, we have:*

$$d_n(\Phi_f) = (d-1)^n - \mu_f(Z). \tag{0.0.1}$$

Let us explain why the assumption that f is square free is important here. Since the singular locus of $F = \mathbb{V}(f)$ is 0-dimensional and f is square free, the polynomials f_i defining Φ_f cannot have a common factor (or else the singular locus of F would have codimension 1), hence the algebraic degree of Φ_f is $d-1$. So let us explain the relation (0.0.1) as follows. As we will see in more details in Example 1.3.9, computing the topological degree of Φ_f is the same as computing the degree of the intersection of n pull back $\mathbb{V}(a_{i0}f_0+\ldots+a_{in}f_n)$ of general hyperplanes $\mathbb{V}(a_{i0}y_0 + \ldots + a_{in}y_n)$ in $\mathbb{P}^n$ after removing the points in the base locus from this intersection. By Bézout's theorem, the intersection $\bigcap_{i=1}^{n} \mathbb{V}(a_{i0}f_0 + \ldots + a_{in}f_n)$ has degree $(d-1)^n$ since the polynomials defining Φ_f have degree $d-1$. To compute the topological degree of Φ_f, it remains to subtract to $(d-1)^n$ the multiplicity of the points in Z. Among all the numbers that can be attached to the base locus of Z, Theorem 10 states that we have actually to subtract the number $\mu_f(Z)$ to $(d-1)^n$.

Example 11. As an example, let us focus on the list of complex homaloidal curves in Theorem 7.

- a smooth conic $F = \mathbb{V}(f) \subset \mathbb{P}^2_{\mathbb{C}}$ verifies $\mu_f = 0$ so $d_t(\Phi_f) = (2-1)^2 = 1$.
- up to a projective change of coordinate, the union of three general lines is the zero locus of $f = x_0x_1x_2$. It has 3 singular points each one verifying $\mu_f(Z,z) = 1$. Hence $d_t(\Phi_f) = (3-1)^2 - 3 = 1$.
- as we saw in Example 9, the global Milnor number of the union of a smooth conic with one of its tangent is equal to 3. Hence $d_t(\Phi_f) = 1$.

From the foundational classification of I.V.Dolgachev of reduced complex homaloidal plane curves, the questions bifurcate in several directions for which we present a non-exhaustive list of results.

Homaloidal hypersurfaces in higher dimension

Over the field $\mathbb{C}$, we saw that there are three types of homaloidal curves in $\mathbb{P}^2$. Concerning the situation in higher dimension, let us first mention that general constructions of homaloidal hypersurfaces existed. For instance, we refer to [Man13, Theorem 21] for general results about the existence of homaloidal hypersurfaces in higher dimension. In addition to the construction of examples, one specific problem was to establish if given $n \geq 3$, there exist homaloidal hypersurfaces of $\mathbb{P}^n$ of any degree. In [CRS08], the three authors answer positively to this question by producing explicit families of homaloidal hypersurfaces of arbitrary degree. More precisely:

Theorem 12. *[CRS08, Theorem 13] For every $n \geq 3$ and for every $d \geq 2n-3$ there exists homaloidal hypersurfaces F of $\mathbb{P}^n$ of degree d.*

Homaloidal hypersurfaces with isolated singularities

In [DP03, 3], A.Dimca and S.Papadima conjectured that given $n \geq 3$, any complex homaloidal hypersurfaces of $\mathbb{P}^n_{\mathbb{C}}$ must have a singular locus of codimension at most $n-1$, i.e. in particular, cannot have a singular locus consisting of isolated points. This conjecture was then proven by J.Huh in [Huh14]. In a recent paper, D.Siersma, J.Steenbrink and M.Tibăr proved that, more generally, there are restrictions to the existence of hypersurfaces with 0-dimensional singular locus and small topological degree. More precisely

Theorem 13. *[SST18, Theorem 1.4] For any integer $k \geq 2$, let K_k denote the set of pairs of integers (n, d) with $n \geq 2$ and $d \geq 3$, such that there exists a projective hypersurface V in $\mathbb{P}^n$ of degree d with isolated singularities and polar degree k.*

Then K_k is finite for any $k \geq 2$.

Homaloidal hypersurfaces in positive characteristic

Another domain concerning homaloidal hypersurfaces is to study the problem over other fields than $\mathbb{C}$ and in particular over algebraically closed fields of positive characteristic. As one can compute, except from a field of characteristic 2, the three plane curves of Theorem 7 are still homaloidal. So let us state the problem as follows:

Problem B. *Over an algebraically closed field* k *of positive characteristic, are there other homaloidal curves than the ones in Dolgachev's classification?*

Another problem in this direction is the following. It was noticed by A.V.Dória, S.H.Hassanzadeh and A.Simis [DHS12] that a common property of the three complex homaloidal curves is that their singular locus Z is a local complete intersection at each of its points. This means that each localisation $\mathcal{O}_{Z,z}$ of the structure sheaf of Z is generated by two elements even if it is generated globally by the three partial derivatives (as we will see, this property is equivalent to the fact that Milnor and Tjurina numbers of the curves coincide).

Problem C. *[DHS12, Question 2.7] Let $f \in \mathrm{k}[x_0, x_1, x_2]$ be a square free homogeneous polynomial whose polar map is birational. Is the singular locus Z of $F = \mathbb{V}(f)$ locally a complete intersection at its points?*

The reduction problem in positive characteristic

In the spirit of studying the difference between characteristic zero and positive characteristic, we also consider the following *reduction problem*. If $f = q_1^{\alpha_1} \dots q_m^{\alpha_m}$ is not square free, or equivalently if F is not reduced, the polar map Φ_f is defined by the mobile part of the linear system generated by $f_0, \dots, f_n$, see the explanation just after Definition 6 about this problem. Over the field of complex numbers, it was established by A.Dimca and S.Papadima [DP03] that Φ_f is birational if and only if so is the polar map $\Phi_{f_{red}}$ associated to $f_{red} = q_1 \dots q_m$. Over a field of positive characteristic, this equivalence trivially fails: in characteristic 2 for $f = x^2yz$, $\Phi_{f_{red}}$ is birational whereas Φ_f is not even dominant. This leads to the following problem.

Problem D. *Over a field of positive characteristic, given Φ_f dominant, is it birational if and only if so is $\Phi_{f_{red}}$?*

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Given a rational map $\Phi : \mathbb{P}_1^n \dashrightarrow \mathbb{P}_2^n$ defined by $n+1$ homogeneous polynomials $\phi_0, \dots, \phi_n \in \mathrm{k}[x_0, \dots, x_n]$ of degree δ without common factor, we consider a locally free presentation of the base ideal $\mathcal{I}_Z$ of Φ:

$$\bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}_1^n}(-a_i) \xrightarrow{M} \mathcal{O}_{\mathbb{P}_1^n}^{n+1} \longrightarrow \mathcal{I}_Z(\delta) \longrightarrow 0 \tag{0.0.2}$$

with $a_i \geq 1$ for all $i \in \{1, \dots, m\}$. It defines an ideal sheaf $\mathcal{I}_{\mathbb{X}}$ on $\mathbb{P}_1^n \times \mathbb{P}_2^n$ generated by the entries of the row matrix $\begin{pmatrix} y_0 & \dots & y_n \end{pmatrix} M$ where $y_0, \dots, y_n$ are the coordinates of $\mathbb{P}_2^n$ ($x_0, \dots, x_n$ being the coordinates of the first factor $\mathbb{P}_1^n$). Actually, the scheme $\mathbb{X} = \mathbb{V}(\mathcal{I}_{\mathbb{X}})$ is the projectivization $\mathbb{P}(\mathcal{I}_Z)$ of the ideal $\mathcal{I}_Z$ embedded in $\mathbb{P}_1^n \times \mathbb{P}_2^n$ and contains the graph Γ of Φ as an irreducible component. We have thus the following commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}_1^n \times \mathbb{P}_2^n \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
 & \mathbb{X} \arrow[u, hook] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & \Gamma \arrow[u, hook] \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\
\mathbb{P}_1^n \arrow[rr, dashed, "\Phi"'] & & \mathbb{P}_2^n
\end{tikzcd}$$

where p_1 (resp. p_2) is the first projection (resp. second projection) and respectively π_1 and σ_1 (resp. π_2 and σ_2) are the restriction of p_1 (resp. p_2) to respectively $\mathbb{X}$ and Γ (resp. $\mathbb{X}$ and Γ).

Our problem in this context is to relate the multidegree of Φ with the geometry of $\mathbb{X}$. As we will explain, this is studying the difference between the symmetric and the Rees algebras of $\mathcal{I}_{\mathbb{X}}$ and, in this perspective, this is a classical problem of commutative algebra, see [Vas05] for an introduction to this point of view or [BCJ09] and [BCS10] for results in this direction.

Chapter 1 is dedicated to explain the background and the framework of the manuscript in a more detailed way and to define the multidegree of rational maps. In **Chapter 2**, we focus more precisely on the definition of the projectivization $\mathbb{P}(\mathcal{I}_Z)$ of an ideal sheaf. We present also geometric properties of $\mathbb{X}$.

In **Chapter 3**, we are especially interested in the resolution of the ideal $\mathcal{I}_{\mathbb{X}}$. Let us explain why. Define $\mathcal{E}$ as the image sheaf of the presentation matrix M in (0.0.2). We call $\mathcal{E}$ the *sheaf of relations* of $\mathcal{I}_Z$. If $\mathcal{E}$ is locally free of rank n then $\mathbb{X}$ is the zero locus of a global section $s \in \mathrm{H}^0\big(\mathbb{P}^n_1 \times \mathbb{P}^n_2, p_1^*(\mathcal{E}^\vee) \otimes p_2^*\mathcal{O}_{\mathbb{P}^n_2}(1)\big)$. This gives to the push forward p_{1*} interesting cohomological properties with respect to a locally free resolution of $\mathcal{I}_{\mathbb{X}}$. For instance, if $\mathcal{E}$ is split as a direct sum of line bundles, then $\mathbb{X}$ is a complete intersection in $\mathbb{P}^n_1 \times \mathbb{P}^n_2$. Hence the ideal sheaf $\mathcal{I}_{\mathbb{X}}$ is resolved by the Koszul complex associated to the generators of $\mathcal{I}_{\mathbb{X}}$ (see Definition 2.2.5 for the definition of the Koszul complex) and, as we will see, this implies that the multidegree of $\mathbb{X}$, which we call the *naive multidegree of* Φ, is computed by the degree of the Chern classes of $\mathcal{E}$. The result in Chapter 3 is that even if $\mathcal{E}$ is not locally free, but assuming instead that $Z = \mathbb{V}(\mathcal{I}_Z)$ is zero-dimensional, $\mathcal{I}_{\mathbb{X}}$ has a locally free resolution closed enough to a Koszul complex, namely:

Proposition 14. *Assuming that $Z = \mathbb{V}(\mathcal{I}_Z)$ is zero-dimensional and denoting $\mathbb{P} = \mathbb{P}^n_1 \times \mathbb{P}^n_2$ and ξ for the first Chern class of $\mathcal{O}_{\mathbb{P}}(0,1)$, a locally free resolution of $\mathcal{I}_{\mathbb{X}}$ reads*

$$0 \to \mathcal{G}_{n+1} \to \mathcal{G}_n \to \ldots \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{I}_{\mathbb{X}} \to 0$$

where $\mathcal{G}_i = \bigoplus_{j=1}^{i} p_1 * \mathcal{T}_{ij} \otimes \mathcal{O}_{\mathbb{P}}(-j\xi)$ when $i \in \{1, \ldots, n\}$ and $\mathcal{G}_{n+1} = p_1^*\mathcal{T}_n \otimes \mathcal{O}_{\mathbb{P}}(-\xi)$ for some locally free sheaves $\mathcal{T}_{ij}$ and $\mathcal{T}_n$ over $\mathbb{P}^n_1$.

Chapter 6 is dedicated to the application of this result, namely, when Z is zero-dimensional, the naive multidegree of Φ is still computed by the length of a cosection of $\mathcal{E}$ (which are the analogues of Chern classes when $\mathcal{E}$ is not locally free).

More generally, **Part II** is dedicated to the applications of considering the projectivization of the base ideal sheaf of a rational map. We emphasize that the dichotomy between the case where $\mathcal{E}$ is locally free and the case it is not locally free is structuring in our work and Part II reflects this dichotomy.

In **Chapter 4** and **Chapter 5**, we focus on the case where $\mathcal{E}$ is locally free. More precisely, in **Chapter 4**, we consider the case where $\mathcal{E}$ is split. As we will explain, this is equivalent to the fact that $\mathcal{I}_Z$ is the ideal of maximal minors of M, so Φ is a determinantal rational map. As we mentioned in Theorem 4, the multidegree of a Cremona map verifies the Cremona inequalities but it is not known if given a sequence $(d_0, \ldots, d_n)$ verifying Cremona inequalities, there exists a Cremona map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$. In Chapter 4, we construct examples of rational map with given multidegree by analysing the scheme $\mathbb{X}$ (or, equivalently by analysing the matrix M). It is in this way that we tackle Problem A. Moreover, we construct

birational maps $\Phi : \mathbb{P}^3_1 \dashrightarrow \mathbb{P}^3_2$ whose inverses Φ^{-1} have the same algebraic degree than Φ. For simplicity we call these maps *n-to-n-tics* by analogy with the more common denomination cubo-cubic and quarto-quartic in degree 3 and 4. Recall from Example 3 that the second multidegree d_2 of Φ is the algebraic degree of Φ^{-1}. Hence n-to-n-tic maps are the maps with multidegree $(1, n, n, 1)$. This work is motivated by the two articles [Pan99] and [DH17]. One of our result in this topic is as follows:

Proposition 15. *Over $\mathbb{P}^3$, let Φ be the determinantal map defined by the 3×3 minors of the following matrix:*

$$M = \begin{pmatrix} x_0 & x_2 + x_3 & x_0^2x_2 + x_0x_1x_2 + x_0x_3^2 \\ 3x_0 + x_1 & x_2 + 2x_3 & x_1^2x_3 + x_1x_2x_3 \\ x_0 + x_1 & x_2 & x_1x_2^2 + x_0x_1x_3 \\ x_0 + 2x_1 & x_3 & x_0^2x_3 + x_1x_3^2 \end{pmatrix}$$

then Φ is a quinto-quintic (i.e. has multidegree $(1, 5, 5, 1)$).

Chapter 5 and Chapter 6 deal with the case where the base locus Z is zero dimensional. In the polar case, in addition to Milnor numbers, we can attach also the *Tjurina number* to a 0-dimensional singularity of a projective hypersurface.

Definition 16. Let Z be the 0-dimensional singular locus of a hypersurface $F = \mathbb{V}(f)$ of $\mathbb{P}^n$ and let $z \in Z$. Via a change of coordinates, suppose that $z = (1 : 0 : \ldots : 0)$.

The *local Tjurina number* at z, denoted by $\tau_f(Z, z)$ is defined as

$$\tau_f(Z, z) = \text{length}\big(\mathcal{O}_{\mathrm{k}^n,z}/(f_\flat, (f_\flat)_1, \ldots, (f_\flat)_n)\big) \quad \text{where } (f_\flat)_i = \frac{\partial f_\flat}{\partial x_i}.$$

The *global Tjurina number* of F, denoted by $\tau_f(Z)$ of F, is the sum $\sum \tau_f(Z, z)$ over all $z \in Z$.

Milnor and Tjurina numbers have very close definitions and given the 0-dimensional singular locus Z of a hypersurface $F = \mathbb{V}(f)$, they verify the inequality

$$\tau_f(Z) \leq \mu_f(Z)$$

but, as we will see, this inequality is strict in general. Anticipating on Chapter 6, we can sum up our work by a study of the number $\mathfrak{d}_n(\Phi_f) = (d-1)^n - \tau_f$ which we call the nth *naive projective degree* or *naive topological degree* and a study of the difference $\mu_f(Z) - \tau_f(Z)$.

In **Chapter 5**, we focus on the case of rational maps $\mathbb{P}^2 \dashrightarrow \mathbb{P}^2$. It is a result following from [Har80] that $\mathcal{E}$ is locally free of rank 2. Let us explain also our initial motivation for studying this problem. Let $f \in \mathrm{k}[x_0, x_1, x_2]$ be a square free polynomial of degree d and let $\mathcal{I}$ be the jacobian ideal sheaf of f, i.e. the ideal sheaf generated by the partial derivatives $f_i = \frac{\partial f}{\partial x_i}$ of f. Letting $\mathcal{E}$ be the sheaf of relations of $\mathcal{I}$, the curve $F = \mathbb{V}(f)$ is *free* if $\mathcal{E}$ is split (following the definition in [Dim15]). If $\mathcal{E}$ is split and $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_2)$ then F is said to be *free of exponent* (d_1, d_2). A result of A.A. du Plessis and C.T.C.Wall in [dPW99] identifies in particular complex curves F of a given degree d with maximal possible global

Tjurina number (see Definition 16 for the definition of Tjurina numbers). These are the free curves of exponents $(1, d-2)$.

But, as we will explain, the data of the global Tjurina number of F is equivalent to the data of the second Chern class of $\mathcal{E}$ via the relation

$$c_2(\mathcal{E}) = (d-1)^2 - \tau_f(Z).$$

Hence the identification of curves with the highest global Tjurina number is equivalent to identifying the curves such that $c_2(\mathcal{E})$ is the smallest possible.

In Definition 2.2.21, we define a generalised Tjurina number for any ideal sheaf $\mathcal{I}$ of $\mathbb{P}^2$ generated by three global sections of $\mathcal{O}_{\mathbb{P}^2}(\delta)$. This is just the length of the scheme $\mathbb{V}(\mathcal{I})$. So for us, a main motivation is to elaborate a similar criterion to split the sheaf of relations $\mathcal{E}$. One result in this chapter is as follows:

Theorem 17. *Let $\mathcal{I}$ be an ideal sheaf over $\mathbb{P}^2$ generated by three global sections ϕ_0, ϕ_1, ϕ_2 of $\mathcal{O}_{\mathbb{P}^2}(\delta)$ and let $\mathcal{E}$ be the sheaf of relations of $\mathcal{I}$ (recall that $\mathcal{E}$ is defined as the kernel of the evaluation map $\mathcal{O}^3_{\mathbb{P}^2} \to \mathcal{I}(\delta)$). Then $-c_1(\mathcal{E}) \leq c_2(\mathcal{E}) + 1$ and equality holds if and only if $\mathcal{E}$ is free of exponents $\big(1, c_2(\mathcal{E})\big)$.*

We emphasize that Theorem 17 is precisely a generalisation of the former result in [dPW99] since we identify free sheaves of exponents $\big(1, c_2(\mathcal{E})\big)$ with sheaves of relations with the smallest second Chern class possible. The second part of this chapter is the classification of the reduced complex plane curves with respect to the second Chern class of their sheaf of relations, that is, we classify curves of degree d and $c_2(\mathcal{E}) = (d-1)^2 - \tau_f$.

In **Chapter 6**, we consider the case where Z is zero-dimensional. In this case $\mathcal{E}$ is not locally free in general. Recall that the nth naive projective degree of Φ is the length of the intersection of n general generators $\sum_{i=0}^{n} \lambda_i \phi_i$ of $\mathcal{I}_Z$ where $\lambda_i \in \mathrm{k}$ for any $i \in \{0, \ldots, n\}$ after removing the points already in Z (see Section 1.3).

Hence the nth projective degree of Φ is the length of the scheme $\mathbb{V}(s)$ defined in the following exact sequence:

$$\mathcal{O}^n_{\mathbb{P}^n_1} \to \mathcal{I}_Z(\delta) \to \mathcal{O}_{\mathbb{V}(s)} \to 0$$

where the morphism $\mathcal{O}^n_{\mathbb{P}^n_1} \to \mathcal{I}_Z(\delta)$ is general. Now consider this morphism in the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}^n_{\mathbb{P}^n_1} \arrow[r, equal] \arrow[d] & \mathcal{O}^n_{\mathbb{P}^n_1} \arrow[d] & \\
0 \arrow[r] & \mathcal{E} = \bigoplus_{i=1}^{m} \mathcal{O}_{\mathbb{P}^n_1}(-a_i) \arrow[r, "M"] \arrow[dr] & \mathcal{O}^{n+1}_{\mathbb{P}^n_1} \arrow[r] \arrow[d] & \mathcal{I}_Z(\delta) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_{\mathbb{P}^n_1} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{V}(s)} \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

So $\mathbb{V}(s)$ is also the support of the cokernel of the map $\mathcal{E} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}_1^n}(-a_i) \to \mathcal{O}_{\mathbb{P}_1^n}$. In other words, the class $[\mathbb{V}(s)]$ of $\mathbb{V}(s)$ in the Chow ring of $\mathbb{P}^n$ is the support of a cosection of $\mathcal{E}$.

The second way to compute this projective degree is to consider the projectivization $\mathbb{X}$ of $\mathcal{I}_Z$ in $\mathbb{P}_1^n \times \mathbb{P}_2^n$ and to compute its decomposition in the Chow ring of $\mathbb{P}_1^n \times \mathbb{P}_2^n$ (see Section 1.2 for these definitions). The problem of this chapter is to establish if those two ways of computing the naive projective degrees, namely considering cosections of $\mathcal{E}$ or the projectivization $\mathbb{X}$, always coincide even if $\mathcal{E}$ is not locally free. We summarize our result as follow, see Theorem 6.1.1 for a more precise statement.

Theorem 18. *In the case when the base locus Z is 0-dimensional, the length of the zero scheme of a cosection of the kernel $\mathcal{E}$ of the evaluation map $\mathcal{O}_{\mathbb{P}_1^n}^{n+1} \to \mathcal{I}_Z(\delta)$ is equal to the nth naive projective degree.*

We answer also positively Problem B and negatively Problem C and Problem D.

Proposition 19. *The curve $F = \mathbb{V}\big((x_1^2 + x_0x_2)x_0(x_1^2 + x_0x_2 + x_0^2)\big)$ is homaloidal if and only if the base field k has characteristic 3, in which case the inverse of the polar map is*

$$\Psi = (x_1^2x_2^2 + x_0x_2^3 + x_2^4 : -x_1^3x_2 - x_0x_1x_2^2 - x_1x_2^3 : -x_1^4 - x_0x_1^2x_2 + x_0x_2^3)$$

Proposition 20. *Let k be an algebraically closed field of characteristic 101.*

(i) The curve $\mathbb{V}\big(z(y^3 + x^2z)\big)$ has polar degree 2 whereas $\mathbb{V}\big(z^{50}(y^3 + x^2z)^{51}\big)$ has polar degree 1.

(ii) The curve $\mathbb{V}\big((y^3+x^2z)(y^2+xz)\big)$ has polar degree 5 whereas the curve $\mathbb{V}\big((y^3+x^2z)^{31}(y^2 + xz)^4\big)$ has polar degree 3.

In **Chapter 7**, the goal is to provide a numerical way of computing the inverse Φ^{-1} of a rational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ using the geometry of the projectivization $\mathbb{X}$ of the base ideal sheaf $\mathcal{I}_Z$ of Φ. We especially focus on finding the inverse of the polar map of $f = (x_1^2 + x_0x_2)x_0(x_1^2 + x_0x_2 + x_0^2)$ in characteristic 3.

Chapter 1

Degrees of rational maps

As we saw in the introduction, different degrees can be attached to a rational map and in particular the projective degrees. In this chapter, before stating our framework about rational maps, we will define these degrees. We end this chapter by a refined presentation of the Cremona group and by answering negatively the following problem (see Subsection 1.4.1 for its motivation).

Problem E. *Is any Cremona map of $\mathbb{P}^2$, of algebraic degree stricly greater than 1, determinantal?*

1.1 Proj of a sheaf

We mostly follow [Har77, II.7] for this background. We only come back to the relative notion of **Proj** of a sheaf of graded $\mathcal{O}_X$-algebras and we refer to [Har77, II.2] for the definition of the Proj of a ring R. The general setting is as follows:

($\triangle$) X is a smooth quasi-projective variety and $\mathcal{S}$ is a positively graded sheaf of graded $\mathcal{O}_X$-module which has a structure of a sheaf of graded $\mathcal{O}_X$-algebras. Thus $\mathcal{S} \simeq \oplus_{d\geq 0}\mathcal{S}_d$ where $\mathcal{S}_d$ is the homogeneous part of degree d. We assume that $\mathcal{S}_0 = \mathcal{O}_X$, that $\mathcal{S}_1$ is a coherent $\mathcal{O}_X$-module, and that $\mathcal{S}$ is locally generated by $\mathcal{S}_1$ as an $\mathcal{O}_X$-algebra.

Construction. For each open subset $U = \mathrm{Spec}(A)$ of X, let $\mathcal{S}_U$ be the graded A-algebra $\Gamma(U, \mathcal{S}_U)$. Then, the schemes $\mathrm{Proj}(\mathcal{S}_U)$ together with their morphisms $\mathrm{Proj}\ \mathcal{S}(U) \to U$ glue to give a scheme $\mathbf{Proj}\,\mathcal{S}$ together with a morphism $\pi : \mathbf{Proj}\,\mathcal{S} \to X$ such that for each open affine $U \subset X$, $\pi^{-1}(U) \simeq \mathrm{Proj}\ \mathcal{S}(U)$. As such, $\mathbf{Proj}\,\mathcal{S}$ has an invertible sheaf $\mathcal{O}(1)$.

Example 1.1.1. Over X, let $\mathcal{S}$ be the polynomial algebra $\mathcal{S} = \mathcal{O}_X[y_0, \ldots, y_n]$. For each open subset $U = \mathrm{Spec}(A)$ of X, $\mathcal{S}_U$ is equal to $A[y_0, \ldots, y_n]$ so $\mathrm{Proj}\ \mathcal{S}_U = \mathbb{P}^n_A$. The morphisms $\mathbb{P}^n_A \to \mathrm{Spec}(A)$ are those associated to the canonical morphisms $A \to A[y_0, \ldots, y_n]$.

The scheme $\mathbf{Proj}\,\mathcal{S}$ is denoted by $\mathbb{P}^n_X$ or $X \times_{\mathrm{k}} \mathbb{P}^n_{\mathrm{k}}$.

Lemma 1.1.2. *[Har77, II.7.9] Let $\mathcal{S}$ be a sheaf of graded algebras on X as in ($\triangle$). Let $\mathcal{L}$ be an invertible sheaf on X, and define a new sheaf of graded algebras*

$\mathcal{S}' = \mathcal{S} \otimes \mathcal{L}$ by $\mathcal{S}'_d = \mathcal{S}_d \otimes \mathcal{L}^d$ for each $d \geq 0$. Then $\mathcal{S}'$ satisfies ($\triangle$) and there is a natural isomorphism $\phi : P' = \mathbf{Proj}\,\mathcal{S}' \to P = \mathbf{Proj}\,\mathcal{S}$, commuting with the projections π and π' to X, and such that

$$\mathcal{O}_{P'}(1) = \phi^*\mathcal{O}_P(1) \otimes \pi'^*\mathcal{L}.$$

Definition 1.1.3. Let X be as in ($\triangle$) and let $\mathcal{G}$ be a coherent sheaf on X. We define the associated *projective space bundle* $\mathbb{P}(\mathcal{G})$ as follows. Let $\mathcal{S} = \mathrm{Sym}(\mathcal{G})$ be the symmetric algebra of $\mathcal{G}$. It verifies $\mathcal{S} = \oplus_{d>0} \mathrm{Sym}_d(\mathcal{G})$. Then $\mathcal{S}$ is a sheaf of $\mathcal{O}_X$-algebras satisfying ($\triangle$) and we define

$$\mathbb{P}(\mathcal{G}) = \mathbf{Proj}\,\mathcal{S}.$$

As such, it comes with a natural projection $\pi : \mathbb{P}(\mathcal{G}) \to X$.

Example 1.1.4. If $\mathcal{G} = \mathcal{O}_X^{n+1}$, then $\mathcal{S} = \mathcal{O}_X[y_0, \ldots, y_n]$ and $\mathbb{P}(\mathcal{G})$ is $\mathbb{P}^n_X$ as in Example 1.1.1. More generally, let $\mathcal{G}$ be a locally free coherent sheaf of rank $n+1$. Given an open set U of X trivialising $\mathcal{G}$, we have that $\pi^{-1}(U) \simeq \mathbb{P}^n_U$ so $\mathbb{P}(\mathcal{G})$ is a "relative projective space" over X.

Proposition 1.1.5. *[Har77, II.7.11] Let X be as in ($\triangle$), $\mathcal{G}$ be a locally free coherent sheaf over X and $\pi : \mathbb{P}(\mathcal{G}) \to X$ be as in Definition 1.1.3. Then:*

(a) *if $\operatorname{rank}\mathcal{G} \geq 2$, there is a canonical isomorphism of graded $\mathcal{O}_X$-algebras $\mathcal{S} \simeq \oplus_{l\in\mathbb{Z}}\pi_*\big(\mathcal{O}(l)\big)$, with the grading on the right hand side given by l. In particular, $\pi_*\big(\mathcal{O}(1)\big) = \mathcal{G}$,*

(b) *there is a natural surjective morphism $\pi^*\mathcal{G} \to \mathcal{O}(1)$.*

1.2 Algebraic cycles and Chow rings

Let k be an algebraically closed field and let X be any variety over k i.e. here integral separated scheme over a field k.

Definition 1.2.1. A *cycle of codimension* r on X is an element of the free abelian group $Z^r(X)$ generated by the closed irreducible subvarieties of X of codimension r. So we write a cycle as $Y = \sum n_i Y_i$ where the Y_i are closed subvarieties of X of codimension r. Sometimes it is usefull to speak of the cycle associated to a closed subscheme. If Z is a closed subscheme of pure codimension r, let $Y_1, \ldots, Y_t$ be those irreducible components of Z which have codimension r, and define the *cycle associated* to Z to be $\sum n_i Y_i$ where n_i is the length of the local ring $\mathcal{O}_{Z,Y_i}$ of the general point u_i of Y_i on Z.

We suppose now that X is a smooth quasi-projective variety of dimension n. We refer to [Har77, II.6 and A.1] for the precise definition of rational equivalence on $Z(X) = \bigoplus_{r=0}^{n} Z^r(X)$.

Definition 1.2.2. For each r, we let $\mathrm{CH}^r(X)$ be the group of cycles of codimension r on X modulo rational equivalence. We denote by $\mathrm{CH}(X)$ the graded group $\oplus_{r=0}^n \mathrm{CH}^r(X)$ and given Z a subvariety of X, we denote by $[Z]$ its class in $\mathrm{CH}(X)$.

Note that $\mathrm{CH}^0(X) = \mathbb{Z}$ and that $\mathrm{CH}^r(X) = 0$ if $r > n$.

We refer to [Har77, Axiom 1 to Axiom 11] and [Har77, Theorem 1.1] for the properties and results making $\mathrm{CH}(X)$ into a graded ring, called the *Chow ring of* X. The product is given by the intersection:

$$\begin{aligned} \mathrm{CH}^i(X) \times \mathrm{CH}^j(X) &\to \mathrm{CH}^{i+j}(X) \\ ([Z_1],[Z_2]) &\longmapsto [Z_1 \cap Z_2] \end{aligned}$$

assuming that $Z_1 \cap Z_2$ has the expected dimension.

Proposition 1.2.3. *[Har77, 2.0.1] Given $n \in \mathbb{N}\backslash\{0\}$, $\mathrm{CH}(\mathbb{P}^n) \simeq \mathbb{Z}[H]/(H^{n+1})$ where H is the class of any hyperplane. In other words, any subvariety of degree d and codimension r in $\mathbb{P}^n$ is rationally equivalent to dH^r.*

Before applying the previous notions for multidegree, let us define the Chern classes following [Har77, 3]

One properties of the intersection product on the Chow ring is as follows:

Property 1.2.4. [Har77, Axiom 11] Let $\mathcal{E}$ be a locally free sheaf of rank r on X, let $\mathbb{P}(\mathcal{E})$ be the associated projective bundle and let $\xi \in \mathrm{CH}^1\left(\mathbb{P}(\mathcal{E})\right)$ be the class of the divisor corresponding to $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$. Let $\pi : \mathbb{P}(\mathcal{E}) \to X$ be the projection. Then π^* makes $\mathrm{CH}\left(\mathbb{P}(\mathcal{E})\right)$ into a free $\mathrm{CH}(X)$-module generated by $1, \xi, \xi^2, \ldots, \xi^{r-1}$.

Definition 1.2.5. Let $\mathcal{E}$ be a locally free sheaf of rank r on a nonsingular quasi-projective variety X. For each $i = 0, 1, \ldots, r$, we define the ith *Chern class* $c_i(\mathcal{E}) \in \mathrm{CH}^i(X)$ by the requirement $c_0(\mathcal{E}) = 1$ and

$$\sum_{i=0}^{r} (-1)^i \pi^* c_i(\mathcal{E}) \xi^{r-i} = 0$$

in $\mathrm{CH}^r\left(\mathbb{P}(\mathcal{E})\right)$.

We refer to [Har77, C1 to C7] for the properties of the Chern classes but we insist on one interpretation that we will use.

Proposition 1.2.6. *[Har77, A.C6] Let $\mathcal{E}$ be a locally free sheaf of rank r on a nonsingular quasi-projective variety X. If the dependency locus W, i.e. the common zero locus, of t global section has codimension $r-t+1$ the class $[W]$ of W is equal to $c_{r-t+1}(\mathcal{E})$ in $\mathrm{CH}^{r-t+1}(\mathcal{E})$.*

1.3 Rational maps

1.3.1 The general setting

Let us first give the following definition extracted from [Har77].

Definition 1.3.1. Let X, Y be varieties. A *rational map* $\Phi : X \to Y$ is an equivalence class of pairs $\langle U, \Phi_U \rangle$ where U is a nonempty open subset of X, Φ_U is a morphism of U to Y, and where $\langle U, \Phi_U \rangle$ and $\langle V, \Phi_V \rangle$ are equivalent if Φ_U and Φ_V agree on $U \cap V$.

When the target variety Y is the projective space $\mathbb{P}^r$ we have the following setup (as a convention, we emphasize that given a vector space V, we denote by $\mathbb{P}(V)$ the space of hyperplanes of V). Let $n = \dim(X)$, $\mathcal{L}$ be a line bundle over X and V be a non-zero vector subspace of $\mathrm{H}^0(X, \mathcal{L})$. We can define a rational map Φ_U on X by sending a point $x \in X$ to the hyperplane $H_x = \{s \in \mathrm{V},\ s(x) = 0\}$ where $s(x)$ is the evaluation of the section s at the point x. Of course, this map is only defined over the points x such that there exists $s \in \mathrm{V}$ verifying $s(x) \neq 0$. The open subset U over which the map Φ_U is defined is precisely the set of these points. Given the basis $(\phi_0, \ldots, \phi_r)$ of V, we see that Φ sends a point $x \in X$ at which it is defined to the point $\big(\phi_0(x) : \ldots : \phi_r(x)\big) \in \mathbb{P}(\mathrm{V}) \simeq \mathbb{P}^r$. Conversely, given a rational map $\Phi : X \dashrightarrow \mathbb{P}^r$, we can find a representative $\langle U, \Phi_U \rangle$ of Φ such that $\mathrm{codim}(X \backslash U) > 1$. Then $\langle U, \Phi_U \rangle$ defines a line bundle $\mathcal{L}_U$ over U extending in a unique way on X (see also [Dol11, 7] for this construction).

Consider the natural evaluation map of sections of $\mathcal{L}$, $\mathrm{ev}' : \mathrm{V} \otimes \mathcal{O}_{\mathrm{X}} \to \mathcal{L}$. It is equivalent to

$$\mathrm{ev} : \mathrm{V} \otimes \mathcal{L}^{\vee} \to \mathcal{O}_{\mathrm{X}}.$$

The image of ev is a sheaf of ideals $\mathcal{I}_Z$ in $\mathcal{O}_X$, called the *base ideal*. Its support $Z = \mathbb{V}(\mathcal{I}_Z)$ is a closed subscheme in X called the *base locus* of Φ. So letting $D_i = \mathbb{V}(\phi_i)$ the scheme of zeros of the section ϕ_i for $i \in \{0, \ldots, r\}$, we have that Z is the scheme theoretic intersection $D_0 \cap \ldots \cap D_r$ in X.

Definition 1.3.2. Let $\Phi : X \dashrightarrow \mathbb{P}^r$ be a rational map with base locus Z and denote $U = X \backslash Z$ so that the representative $\Phi_U : U \to \mathbb{P}^r$ of Φ is a morphism. The closure of Φ_U in $\mathbb{P}^r$ does not depend on the representative of Φ and is called the *image of* Φ, denoted by $\mathrm{Im}(\Phi)$. Given also a closed subvariety W of $\mathbb{P}^r$, the closure in X of the inverse image $\Phi_U^{-1}(W)$, denoted by $\Phi^{-1}(W)$ is called the *inverse transform* of W under Φ.

Definition 1.3.3. Let $\Phi : X \dashrightarrow \mathbb{P}^r$ be a rational map and let Y be the image variety of Φ. Denoting Z the base locus of Φ and $U = X \backslash Z$, the *graph* Γ of Φ is defined as the closure

$$\overline{\{\big(x, \Phi(x)\big),\ x \in U\}} \subset X \times Y$$

of the graph of the morphism $\Phi_U : U \to Y$ in $X \times Y$ (it does not depend on the representative of Φ).

Since dominant rational maps $\Phi : X \dashrightarrow Y$ and $\Psi : Y \dashrightarrow X$ can be composed, we are especially interested in the case when this composition has the identity map as representative.

Definition 1.3.4. Let X be a quasi-projective variety over a field k and let $\Phi : X \dashrightarrow \mathbb{P}^r$ be a rational map with its image variety $Y \subset \mathbb{P}^r$. Denoting $\mathrm{k}(X)$ (resp. $\mathrm{k}(Y)$) the field of fraction of X (resp. Y), Φ defines a morphism $\Phi^* : \mathrm{k}(Y) \to \mathrm{k}(X)$ and Φ is *birational* if Φ^* is an isomorphism. This is equivalent to the fact there exist a rational map $\Psi : Y \dashrightarrow X$ such that the compositions $\Phi \circ \Psi = id_Y$ and $\Psi \circ \Phi = id_X$ as rational maps.

1.3.2 Construction with homogeneous polynomials

Our main source of examples and applications come from the situation where $X = \mathbb{P}^n$ and $r = n$. In this case, every line bundle $\mathcal{L}$ is isomorphic to $\mathcal{O}_{\mathbb{P}^n}(\delta)$ for an integer δ where $\mathcal{O}_{\mathbb{P}^n}(\delta)$ is a line bundle with the property that $\mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(\delta)\right)$ is isomorphic to the space $\mathrm{k}[x_0, \cdots, x_n]_\delta$ of homogeneous polynomials in $n+1$ variables of degree δ. Hence we can identify the sections $\phi_i \in \mathrm{H}^0(\mathcal{L}, \mathcal{O}_{\mathbb{P}^n}(\delta))$ with homogeneous polynomials of degree δ. Now, letting V be an $n+1$ dimensional subspace of $\mathrm{H}^0(\mathcal{L}, \mathcal{O}_{\mathbb{P}^n}(\delta))$, a rational map Φ between two projective spaces $\mathbb{P}^n$ and $\mathbb{P}^r$ over a field k, denoted $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, is the data of $n+1$ homogeneous polynomials $\phi_0, \ldots, \phi_n \in \mathrm{k}[x_0, \cdots, x_n]$ of the same degree δ without common component.

Given $\Phi : \mathbb{P}^n_1 \dashrightarrow \mathbb{P}^n_2$ and another rational map $\Psi : \mathbb{P}^n_2 \dashrightarrow \mathbb{P}^n_3$ defined by $n+1$ homogeneous polynomials $\psi_0, \ldots, \psi_n \in \mathrm{k}[y_0, \ldots, y_n]$ such that $\mathrm{Im}(\Phi)$ is not contained in the base locus of Ψ (this is the case for instance if Φ is dominant), the *composition* $\Psi \circ \Phi$ of Φ and Ψ is the map from $\mathbb{P}^n_1$ to $\mathbb{P}^n_3$ defined by the polynomial $\psi_0(\phi_0, \ldots, \phi_n), \ldots,\ \psi_n(\phi_0, \ldots, \phi_n)$ where we substitute the variables y_i by the polynomials ϕ_i.

We saw in Example 1 that there a preferred representative of rational map of $\mathbb{P}^n$, this motivates the following comment.

Remark 1.3.5. Given a rational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, we may always consider that the map is defined by polynomials $\phi_0, \ldots, \phi_n$ without common factor. This is requiring that the codimension of the base locus $Z = \{\phi_0 = \ldots = \phi_n = 0\}$ is greater than 1.

1.3.3 Multidegree of a rational map and cycles in a Segre product

We present first the *topological degree*. Let $\Phi : X \dashrightarrow Y$ be a dominant rational map between two irreducible varieties X and Y of the same dimension over an algebraically closed field k. This situation corresponds to a field extension $\Phi^* : \mathrm{k}(Y) \to \mathrm{k}(X)$ between the respective fraction fields of X and Y.

Definition 1.3.6. The *topological degree* of Φ is the degree

$$d_t(\Phi) = [\mathrm{k}(X) : \mathrm{k}(Y)]$$

of the field extension of $\mathrm{k}(X)$ over $\mathrm{k}(Y)$.

From Definition 1.3.4, we have the following relation between topological degree and rationality.

Proposition 1.3.7. *The map Φ is birational if and only if $d_t(\Phi) = 1$.*

Given that $\Phi : X \dashrightarrow Y$ is dominant between two varieties of the same dimension over an algebraically closed field k, we explain now a geometric interpretation of the topological degree following [Har92, 7.16].

Proposition 1.3.8. *The topological degree $d_t(\Phi)$ of Φ is equal to the number of points in a general fibre of Φ, i.e. the inverse transform of a general point of Y under Φ.*

Proof. First we can assume that X and Y are affine open subsets since the content of the proposition is local. So we can assume that $X \subset \mathbb{A}^n$ and $Y \subset \mathbb{A}^m$ and Φ is the projection from the graph $\Gamma \subset \mathbb{A}^n \times \mathbb{A}^m$ of Φ on Y which is the restriction of a linear projection $\mathbb{A}^n \times \mathbb{A}^m \to \mathbb{A}^m$ to Γ.

It is thus enough to prove the claim for a map $\Phi : X \dashrightarrow Y$ of affine varieties given as the restriction of the projection

$$\begin{array}{rcl} p: & \mathbb{A}^n & \dashrightarrow \mathbb{A}^{n-1} \\ & (x_1, \ldots, x_n) & \dashrightarrow (x_1, \ldots, x_{n-1}) \end{array}$$

In this case the fraction field $\mathrm{k}(X)$ of X is generated over $\mathrm{k}(Y)$ by the element x_n. Since X and Y have the same dimension and Φ is dominant, x_n is algebraic over $\mathrm{k}(Y)$. Thus, let $a_i \in \mathrm{k}(Y)$ for $i \in \{0, \ldots, d\}$ such that

$$G(x_1, \ldots, x_n) = a_0(x_1, \ldots, x_{n-1})x_n^d + a_1(x_1, \ldots, x_{n-1})x_n^{d-1} + \ldots$$

is the minimal polynomial satisfied by x_n (so $[\mathrm{k}(X) : \mathrm{k}(Y)] = d$). After clearing denominators, we may take the a_i to be regular functions on Y i.e. polynomials in $x_1, \ldots, x_{n-1}$.

Let $\Delta(x_1, \ldots, x_{n-1})$ be the discriminant of G as a polynomial in x_n. Since G is reduced in $\mathrm{k}(Y)[x_n]$ and k is algebraically closed, Δ cannot vanish identically on Y. It follows that the loci $\{a_0 = 0\}$ and $\{\Delta = 0\}$ are proper subvarieties of Y and, on the complement of their union, the fibres of Φ consist of exactly d points. $\square$

Example 1.3.9. To illustrate Proposition 1.3.8 we explain the following computational way to determine the topological degree of a rational map $\Phi : \mathbb{P}_1^n \dashrightarrow \mathbb{P}_2^n$ given by $n+1$ homogeneous polynomials $\phi_0, \ldots, \phi_n$ in $n+1$ variables each one of degree δ and such that $\dim \mathbb{V}(\phi_0, \ldots, \phi_n) \leq n-2$. Here we let $\mathbb{P}_1^n$ and $\mathbb{P}_2^n$ be two projective spaces of dimension n over k with respective coordinates $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$.

(1) The first step is to take a general point in $\mathbb{P}_2^n$. By Bézout's theorem, this is the same as the choice of n general hyperplanes $H_1, \ldots, H_n$ in $\mathbb{P}_2^n$ intersecting precisely at this point. In the following, we let:

$$H_i \ : \ a_{i0}y_0 + \ldots + a_{in}y_n = 0$$

be the equations of the hyperplanes H_i. Computationally, one often works with a finite fields or even prime fields $\mathbb{Z}/p\mathbb{Z}$. Under many points of view, this is a good approximation of infinite fields and even of algebraically closed fields if the prime number p is big enough. Over finite fields, "random" means that all elements of the field have the same probability of being chosen.

(2) It remains to compute the fibre $\mathbb{F}_y$ of the point y in the intersection $\cap_i H_i$. Let us remark that the fibre $\{x \in \mathbb{P}_1^n, \ \Phi(x) = y\}$ of y is contained in the intersection $\cap_i \Phi^* H_i$ where $\Phi^* H_i$ are hypersurfaces of $\mathbb{P}_1^n$ given by the equations:

$$\Phi^* H_i \ : \ a_{i0}\phi_0 + \ldots + a_{in}\phi_n = 0.$$

Actually, if for any $x \in \mathbb{P}^n_1$ there exists $i \in \{0, \dots, n\}$ such that $\phi_i(x) \neq 0$ then x is in the fibre of y. This is just saying that $\Phi(x)$ is well defined (x is not in the base base locus of Φ) and that y is the only point in the intersection $\cap_i H_i$.

(3) The fibre of y is in general strictly contained in $\cap_i \Phi^* H_i$ since the latter scheme contains also the base locus $Z = \mathbb{V}(\phi_0, \dots, \phi_n)$ of Φ. The computational step consisting in removing Z from $\cap_i \Phi^* H_i$ is the *saturation*

$$I_{\mathbb{F}_y} = [I_H : I_Z^\infty] = \bigcup_{i \geq 1} [I_H : I_Z^i]$$

where I_H is the ideal of $\cap_i \Phi^* H_i$, I_Z is the ideal of Z and

$$[I_H : I_Z^i] = \{r \in \mathrm{k}[x_0, \dots, x_n],\ \exists f \in I_Z^i,\ rf \in I_H\}$$

is the *ideal quotient* of I_H by I_Z^i [Eis95, Page 15]. Indeed the saturation of an ideal I with respect to an ideal J corresponds to consider all the primary components of I which are not contained in any infinitesimal neighbourhood of $\mathbb{V}(J)$. We refer to [CLO07, 4.7 Primary Decomposition of Ideals] for the definition of primary components of an ideal and [CLO07, 4.4 Zariski Closure and Quotients of Ideals, Exercices 8 to 10] for the geometric meaning of the saturation and its actual computation via Groebner basis algorithms. The ideal $I_{\mathbb{F}_y}$ is then the ideal of the fibre $\mathbb{F}_y$.

Now, we turn to the more general notion of multidegree of a rational map. Let X be a smooth irreducible subvariety of $\mathbb{P}^n$, Y be a smooth irreducible subvariety of $\mathbb{P}^m$ and $\Phi : X \dashrightarrow Y$ be a rational map from X to Y. We denote by Γ the graph of Φ. The situation is summed up with the following commutative diagram:

$$\begin{tikzcd}
 & \mathbb{P}^n \times \mathbb{P}^m \arrow[dl, "p_1"'] \arrow[dr, "p_2"] \arrow[d, hook, "\Gamma"] & \\
\mathbb{P}^n \arrow[ur, "\sigma_1"] \arrow[rr, dashed, "\Phi"'] & & \mathbb{P}^m \arrow[ul, "\sigma_2"']
\end{tikzcd}$$

where p_1 (resp. p_2) is the first (resp. second) projection.

Proposition 1.3.10. *[Dol00, 7.1.3] The Chow group of $\mathbb{P}^n \times \mathbb{P}^m$ is the tensor product of the Chow group of $\mathbb{P}^n$ with the Chow group of $\mathbb{P}^m$.*

$$\bigoplus_{i,j \geq 0} \mathrm{CH}^i(\mathbb{P}^n) \otimes \mathrm{CH}^j(\mathbb{P}^m) \simeq \bigoplus_{i=0}^{n} \bigoplus_{j=0}^{m} \mathbb{Z} \otimes \mathbb{Z},$$

Given $i \in \{1, 2\}$, we denote by h_i the cycle class of $p_i^*(H_i)$ in $\mathrm{CH}(\mathbb{P}^n \times \mathbb{P}^m)$ where H_1 (resp. H_2) is the class of a hyperplane in $\mathbb{P}^n$ (resp. $\mathbb{P}^m$). Each cycle of $\mathbb{P}^n \times \mathbb{P}^m$ decomposes into a sum $\sum_{i=0}^{n} \sum_{j=0}^{m} d_{ij} h_1^i h_2^j$.

Definition 1.3.11. The *multidegree* of a subvariety G of dimension k in $\mathbb{P}^n \times \mathbb{P}^m$ is the sequence of coefficients $(d_0, \ldots, d_k)$ in the decomposition of $[G] = \sum_{i=0}^{k} d_i h_1^i h_2^{k-i}$.

Definition 1.3.12. The *multidegree* of a rational map $\Phi : X \dashrightarrow Y$ with $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$ is the multidegree of the graph Γ of Φ in $\mathbb{P}^n \times \mathbb{P}^m$. Since $\dim(\Gamma) = \dim(X)$, we decompose $[\Gamma]$ as follows:

$$[\Gamma] = \sum_{i=0}^{k} d_i h_1^i h_2^{k-i}.$$

The coefficient d_i is called the ith *projective degree* of Φ .

Remark 1.3.13. As defined in the generality of Definition 1.3.12, the nth projective degree $d_n(\Phi)$ of a rational map Φ does not recover completely the first definition of topological as degree of the general fibre and we illustrate why in the following example.

Let $\mathcal{C}$ be the image in k^3 of the map

$$\begin{aligned} \phi : \quad & \mathrm{k} \longrightarrow \mathrm{k}^3 \\ & x_0 \mapsto (x_0, x_0^2, x_0^3) \end{aligned}$$

$\mathcal{C}$ is actually a smooth rational cubic curve usually called *a twisted cubic*. In homogeneous coordinates, $\mathcal{C}$ is the image of the map

$$\begin{aligned} \Phi : \quad & \mathbb{P}^1 \longrightarrow \mathbb{P}^3 \\ & (x_0 : x_1) \mapsto (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3) \end{aligned}$$

This latter map is birational onto its image $\mathcal{C}$ so $d_t(\Phi) = 1$ with respect to Definition 1.3.6. However the cycle class of the graph Γ_Φ of Φ in the Chow ring of $\mathbb{P}^1 \times \mathbb{P}^3$ is:

$$[\Gamma_\Phi] = (h_1 + h_2)^3 = 3h_1 h_2^2 + h_2^3$$

where $h_1 = p_1^* H_1$ (resp. $h_2 = p_2^* H_1$) is the cycle class of the pull back of a hyperplane of $\mathbb{P}^1$ (resp. $\mathbb{P}^3$) and where $p_1 : \mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^1$ (resp. $p_2 : \mathbb{P}^1 \times \mathbb{P}^3 \to \mathbb{P}^3$) is the first projection (resp. second). Hence $d_1(\Phi) = 3$, the degree of $\mathcal{C}$ in $\mathbb{P}^3$. For more precision about this justification and the cycle class of a variety in $\mathbb{P}^1 \times \mathbb{P}^3$ we refer to Section 1.2.

Our justification to maintain Definition 1.3.12 is that our results concern dominant rational maps $\Phi : X \dashrightarrow \mathbb{P}^n$ where the definition of nth *projective degree* and *topological degree* coincide.

More generally, with the previous notation, suppose that Φ is dominant over Y and that $\dim(X) = \dim(Y) = k$. Then

Proposition 1.3.14. *Let $\Phi : X \dashrightarrow Y$ be a rational map and let fix the embedding $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$. Then*

$$d_k(\Phi) = \deg(Y) d_t(\Phi).$$

i.e. the topological degree $d_t(\Phi)$ of Φ is the kth projective degree of Φ divided by the degree of Y in $\mathbb{P}^m$.

Remark 1.3.15. With the notation as in Definition 1.3.12, let $i \in \{0, \dots, k\}$ and let H_1^i be a general $(n-i)$-plane of $\mathbb{P}_1^n$, H_2^{k-i} be a general $(m-k+1)$-plane of $\mathbb{P}_2^m$. Then if the intersection $H_1^i \cap \Phi^{-1}(H_2^{k-i})$ is reduced and 0-dimensional, the ith projective degree $d_i(\Phi)$ of Φ is:

$$d_i(\Phi) = \operatorname{card}\big(H_1^i \cap \Phi^{-1}(H_2^{k-i})\big)$$

where, by convention, we set $H_1^0 = \mathbb{P}^n$ and $H_2^0 = \mathbb{P}^m$.

1.4 The Cremona group

Definition 1.4.1. Given an integer $n \geq 1$, a rational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ of topological degree $d_n(\Phi) = 1$ is called a *Cremona map*.

Definition 1.4.2. Given $n \geq 1$, the set of all Cremona maps of $\mathbb{P}^n$ has a structure of group with respect to the composition law of rational maps. We call this group the *Cremona group of* $\mathbb{P}^n$, denoted by $\operatorname{Bir}(\mathbb{P}^n)$.

1.4.1 First properties of the Cremona group

There are a lot of questions and results about the Cremona group. We restrict our presentation to the foundational result of Noether and Castelnuovo establishing that the Cremona group $\operatorname{Bir}(\mathbb{P}^2_{\mathbb{C}})$ over the complex field has a remarkable set of generators. We define first the *projective automorphisms of* $\mathbb{P}^n$.

Definition 1.4.3. Let $n \geq 1$, to any matrix $A = (a_{ij})_{0 \leq i \leq n, 0 \leq j \leq n} \in \mathrm{PGl}_{n+1}$, we can associate a birational map sending $(x_0 : \dots : x_n) \in \mathbb{P}^n$ to $(\sum_{j=0}^{n} a_{0j} x_j : \dots : \sum_{j=0}^{n} a_{nj} x_j)$, the inverse being the map associated to A^{-1}. Such a map is called a *projective automorphisms of* $\mathbb{P}^n$.

Theorem 1.4.4. *(Castelnuovo-Noether's theorem) The Cremona group* $\operatorname{Bir}(\mathbb{P}^2_{\mathbb{C}})$ *is generated by the projective automorphisms of* $\mathbb{P}^2$ *and the standard Cremona map* $\tau = (x_1x_2 : x_0x_2 : x_0x_1)$

We refer to [Dol11, 7.5] for a proof of this result.

Example 1.4.5. Let

$$\begin{aligned} \Phi : \quad & \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \\ & (x_0 : x_1 : x_2) \dashrightarrow (x_0^2 : x_0x_1 : x_1^2 - x_0x_2). \end{aligned}$$

It is an involution, i.e. $\Phi \circ \Phi = id$ and we can compute that

$$\Phi = A_4 \circ \tau \circ A_3 \circ \tau \circ A_2 \circ \tau \circ A_1$$

where $A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & -2 \\ 1 & 1 & 0 \end{pmatrix}$, $A_3 = \begin{pmatrix} 4 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $A_2 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$,
$A_1 = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ and $\tau = (x_1x_2 : x_0x_2 : x_0x_1)$.

An output of Theorem 1.4.4 is that $\mathrm{Bir}(\mathbb{P}^2_{\mathbb{C}})$ has a rather simple set of generators. This no longer true when $n \geq 3$ as stated by the following result.

Theorem 1.4.6. *[Pan99] When $n \geq 3$, the Cremona group $\mathrm{Bir}(\mathbb{P}^n_{\mathbb{C}})$ cannot be generated by a set of Cremona maps of bounded degree.*

Determinantal rational maps

Let $n \geq 1$ and M be a matrix of size $(n+1) \times n$ whose entries are homogeneous polynomials in $\mathrm{k}[x_0, \cdots, x_n]$ and such that all the entries of one given column have the same algebraic degree. Then the $n \times n$-minors of M define a rational map of $\mathbb{P}^n$.

Definition 1.4.7. Given $n \geq 1$ and $M \in \mathrm{Mat}_{n+1,n}(\mathrm{k}[x_0, \cdots, x_n])$ with homogeneous entries, a rational map defined by the $n \times n$-minors of M is called a *determinantal rational map*. If this map is birational we call it a *determinantal Cremona map*.

Example 1.4.8. Over $\mathrm{k}[x_0, x_1, x_2]$, let Φ be the determinantal rational map defined by the 2×2-minors of

$$M = \begin{pmatrix} x_0 & x_1x_2 \\ x_1 & x_0^2 \\ 0 & x_0x_2 \end{pmatrix}.$$

That is, the map Φ is represented by the polynomials

$$(x_0^3 - x_1^2x_2, x_0^2x_2, x_1x_0x_2)$$

and is actually a Cremona map of $\mathbb{P}^2$ whose base locus Z is supported over the points $(0:0:1)$ and $(0:1:0)$. Its inverse Φ^{-1} is defined by the polynomials

$$(-x_0x_1^2 - x_1x_2^2, -x_0x_1x_2 - x_2^3, -x_1^3)$$

which are the 2×2-minors of the matrix

$$M' = \begin{pmatrix} -x_2 & -x_1^2 \\ x_1 & 0 \\ 0 & x_0x_1 + x_2^2 \end{pmatrix}.$$

In $\mathbb{P}^2$, it is remarkable that all Cremona maps of algebraic degree 2 are actually determinantal. Such a result follows from the classical description of Cremona map of algebraic degree 2 (see [Dés12]). When looking at a list of Cremona maps of algebraic degree 3 such as in [Dés12, 4.6], we compute that the first members

of the list are also determinantal. Since the standard Cremona map τ is itself determinantal as the 2×2-minors of the matrix $\begin{pmatrix} x_0 & x_0 \\ 0 & x_1 \\ -x_2 & 0 \end{pmatrix}$ and since τ is the only generator of $\mathrm{Bir}(\mathbb{P}^2)$ of algebraic degree not equal to 1, we end up with Problem E.

The answer is negative as illustrated by the following map Φ in Proposition 1.4.9 which is the composition the map defined by the 2×2-minors of the matrix

$$\begin{pmatrix} x_1x_2 & x_0x_2 \\ x_0x_2 & x_0x_1 \\ x_0x_1 & x_1x_2 \end{pmatrix}$$

and the standard Cremona map $\tau = (x_1x_2, x_0x_2, x_0x_1)$.

Proposition 1.4.9. *The map $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ defined by the polynomials*

$$\begin{cases} \phi_0 = x_0^3x_1^2 - x_0x_1^3x_2 - x_0^2x_1x_2^2 + x_1^2x_2^3 \\ \phi_1 = -x_0^2x_1^3 + x_0^3x_1x_2 + x_0x_1^2x_2^2 - x_0^2x_2^3 \\ \phi_2 = -x_0^2x_1^2x_2 + x_0^3x_2^2 + x_1^3x_2^2 - x_0x_1x_2^3 \end{cases}$$

is a Cremona map of $\mathbb{P}^2$ which is not determinantal.

Proof. We introduce several tools that we will use in the next chapters. Over the coordinate ring $R = \mathrm{k}[x_0, x_1, x_2]$ of $\mathbb{P}^2$, we let $I_Z = (\phi_0, \phi_1, \phi_2)$. Of course the base ideal sheaf $\mathcal{I}_Z$ of Φ defined in the first section is just the sheafification of I_Z. Now consider a minimal free presentation of I_Z

$$F \xrightarrow{M} R^3 \longrightarrow I_Z \longrightarrow 0 \tag{1.4.1}$$

where F is a free R-module and M is the presentation matrix of I_Z. Since the base locus Z of Φ is assumed to be of codimension strictly greater than 1, the Hilbert-Burch theorem [Eis95, 20.15] states precisely that $\mathrm{rank}(F) = 2$ if and only if $\mathcal{I}_Z$ is generated by the 2×2-minors of M. But in our case, a minimal free presentation of I_Z actually reads

$$R^3 \xrightarrow{M} R^3 \longrightarrow I_Z \longrightarrow 0 \tag{1.4.2}$$

and

$$M = \begin{pmatrix} x_0x_1^2 - x_0^2x_2 & x_1^2x_2 - x_0x_2^2 & 0 \\ x_0^2x_1 - x_1^2x_2 & 0 & -x_0^2x_2 + x_1x_2^2 \\ 0 & x_0x_1^2 - x_1x_2^2 & x_0^2x_1 - x_0x_2^2 \end{pmatrix}.$$

Hence Φ is not determinantal and it is a computation, for example using the algorithm we explained in Example 1.3.9 that Φ is birational.

```
i1 : k = QQ

o1 = QQ

o1 : Ring
```

```
i2 : R = k[x_0..x_2]

o2 = R

o2 : PolynomialRing

i3 :
I = ideal(x_0^3*x_1^2-x_0*x_1^3*x_2-x_0^2*x_1*x_2^2+x_1^2*x_2^3,
-x_0^2*x_1^3+x_0^3*x_1*x_2+x_0*x_1^2*x_2^2-x_0^2*x_2^3,
-x_0^2*x_1^2*x_2+x_0^3*x_2^2+x_1^3*x_2^2-x_0*x_1*x_2^3);

o3 : Ideal of R

i4 : J = ideal ( (gens I)*random(R^{3:1},R^{2:1}) );

o4 : Ideal of R

i5 : degree saturate(J,I)

o5 = 1
```

hence the fibre of a general point has cardinal 1 so the topological degree of Φ is 1. $\square$

Remark 1.4.10. Let us explain why we restrict Problem E to birational maps of $\mathbb{P}^2$ of degree stricly greater than 1. It is because the minimal free resolution of the base ideal I_Z of any projective automorphism of $\mathbb{P}^2$ reads

$$0 \longrightarrow R \longrightarrow R^3 \longrightarrow R^3 \longrightarrow I_Z \longrightarrow 0.$$

Indeed, Z beeing empty, the three generators of $\mathcal{I}_Z$ are a regular sequence (see Definition 2.2.2) so I_Z is resolved by the Koszul complex associated to the three generators of $\mathcal{I}_Z$ (see Definition 2.2.5). Hence any projective automorphism of $\mathbb{P}^2$ is not determinantal.

Part I

Projectivization of an ideal sheaf

Chapter 2

Torsion of the symmetric algebra

In this chapter, we consider a smooth quasi-projective variety X. Recall from Subsection 1.3.1 that a rational map $\Phi : X \dashrightarrow \mathbb{P}^n$ defines an ideal sheaf $\mathcal{I}_Z$ which is the image of the evaluation morphism $\mathrm{V} \otimes \mathcal{O}_X \to \mathcal{L}$, where $\mathcal{L}$ is a line bundle over X and V is the subspace of global sections of $\mathcal{L}$ defining Φ.

The main idea here is to consider the projectivization $\mathbb{P}(\mathcal{I}_Z)$ of $\mathcal{I}_Z$. A locally free presentation of $\mathcal{I}_Z$ determines an embedding of $\mathbb{P}(\mathcal{I}_Z)$ in $\mathbb{P}^n_X$. Since $\mathbb{P}(\mathcal{I}_Z)$ contains also the blow-up $\tilde{X}$ of X along $Z = \mathbb{V}(\mathcal{I}_Z)$, the birationality of Φ and, more generally, the multidegree of Φ, defined over $\tilde{X}$, can be studied in terms of the properties of $\mathbb{P}(\mathcal{I}_Z)$ (see Subsection 1.3.3 for the definition of the multidegree). A *locally free presentation* of an ideal sheaf $\mathcal{I}_Z$ generated by $n+1$ global sections $(\phi_0, \ldots, \phi_n)$ of a line bundle $\mathcal{L}$ is the data of an exact sequence:

$$\mathcal{F} \xrightarrow{M} \mathrm{V} \otimes \mathcal{O}_X \xrightarrow{\begin{pmatrix}\phi_0 & \cdots & \phi_n\end{pmatrix}} \mathcal{I}_Z \otimes \mathcal{L} \longrightarrow 0$$

where $\mathcal{F}$ is locally free. The matrix M is called a *presentation matrix* of $\mathcal{I}_Z$. Since our applications concern mostly rational maps $\Phi : \mathbb{P}^n_1 \dashrightarrow \mathbb{P}^n_2$, we emphasize that, in this case, we identify the sections ϕ_i with their corresponding homogeneous polynomials in $R = \mathrm{k}[x_0, \cdots, x_n]$ (with $\mathbb{P}^n_1 = \mathrm{Proj}(R)$) of the same degree δ and that the ideal sheaf $\mathcal{I}_Z$ is then the sheafification of the ideal I_Z generated by polynomials $\phi_0, \ldots, \phi_n$ over R. With this identification a presentation of $\mathcal{I}_Z$ is then the sheafification of a presentation of I_Z

$$F \xrightarrow{M} R^{n+1} \xrightarrow{\begin{pmatrix}\phi_0 & \cdots & \phi_n\end{pmatrix}} I_Z \longrightarrow 0$$

where F is a free R-module. In this perspective, we can apply results from commutative algebra and eventually make the computation with a computer algebra system such as Macaulay2 to compute the free presentation of I_Z and to infer a presentation of $\mathcal{I}_Z$. As a matter of notation, we denote by M both a matrix presentation of I_Z and the sheafified map $\tilde{M}$ of M. The justification for this simplification is that the entries of M can be identified with their corresponding global sections which determine the map $\mathcal{F} \xrightarrow{M} \mathrm{V} \otimes \mathcal{O}_{\mathbb{P}^n_1}$ where $\mathcal{F}$ is the sheafification of F.

2.1 Projectivization and blow-up

In the following, we will use that *the formation of the symmetric algebra commutes with base change*. Let us explain now what we mean by that.

Proposition 2.1.1. *[Eis95, Proposition A.2.2] Let R be a ring, R' be an R-algebra and M be an R-module. Then there is a R-module isomorphism*

$$\mathrm{Sym}(M \otimes_R R') \simeq \mathrm{Sym}(M) \otimes_R R'.$$

In our context, from a relative point of view, this implies that the localization of the symmetric algebra of an ideal sheaf $\mathcal{I}_Z$ is the symmetric algebra of the localization of $\mathcal{I}_Z$.

Notation 2.1.2. Letting X and $\mathcal{G}$ be as in Definition 1.1.3 and given a subscheme $\mathbb{L}$ of $\mathbb{P}(\mathcal{G})$ and any $x \in X$, we denote by $\mathbb{L}_x$ the scheme-theoretic fibre of the projection $\pi : \mathbb{P}(\mathcal{G}) \to X$ restricted to $\mathbb{L}$ above x.

2.1.1 Projectivization of an ideal sheaf

We consider now the special case of the projectivization of an ideal sheaf. Let X be a smooth quasi-projective variety, let $\mathcal{I}$ be an ideal sheaf over X and let

$$\mathcal{O}_X^{r+1} \xrightarrow{(\phi_0 \,\dots\, \phi_r)} \mathcal{I} \otimes \mathcal{L} \longrightarrow 0, \tag{2.1.1}$$

be a surjection for some ample line bundle $\mathcal{L}$ on X. We have an embedding of $\mathbb{P}(\mathcal{I} \otimes \mathcal{L})$ in $\mathbb{P}^r_X$. Indeed, by [Bou70, A.III.69.4], the surjection $\mathcal{O}_X^{r+1} \to \mathcal{I} \otimes \mathcal{L} \to 0$ of (2.1.1) implies that $\mathrm{Sym}(\mathcal{O}_X^{r+1})$ surjects over $\mathrm{Sym}(\mathcal{I} \otimes \mathcal{L})$ and that the kernel of this latter surjection is generated by the kernel of $\mathcal{O}_X^{r+1} \to \mathcal{I} \otimes \mathcal{L}$. Hence denoting $\mathcal{F}$ the kernel of $(\phi_0 \,\dots\, \phi_r)$ in (2.1.1) and M the morphism $\mathcal{F} \to \mathcal{O}_X^{r+1}$, we have that the ideal sheaf $\mathcal{I}_{\mathbb{P}(\mathcal{I})}$ of $\mathbb{P}(\mathcal{I})$ is locally generated by the entries of the row matrix $\begin{pmatrix} y_0 & \dots & y_r \end{pmatrix} p^* M$ where $y_0, \dots, y_r$ are the relative homogeneous coordinates of $\mathbb{P}^r_X$ and $p : \mathbb{P}^r_X \to X$ is the projection.

Now by Lemma 1.1.2, we have that $\mathbb{P}(\mathcal{I}_Z) \simeq \mathbb{P}(\mathcal{I}_Z \otimes \mathcal{L})$ and moreover that:

$$\mathcal{O}_{\mathbb{P}(\mathcal{I}_Z)}(1) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{I}_Z \otimes \mathcal{L})}(1) \otimes p^* \mathcal{L}^\vee.$$

So we consider that $\mathbb{P}(\mathcal{I}_Z)$ itself is embedded in $\mathbb{P}^r_X$. Let us summarize these properties with the following proposition.

Proposition 2.1.3. *Let X be a smooth quasi-projective variety, $\mathcal{L}$ be an ample line bundle on X and let $\mathcal{I}$ be an ideal sheaf of X. A locally free presentation of $\mathcal{I} \otimes \mathcal{L}$,*

$$\mathcal{F} \xrightarrow{M} \mathcal{O}_X^{r+1} \xrightarrow{(\phi_0 \,\dots\, \phi_r)} \mathcal{I} \otimes \mathcal{L} \longrightarrow 0 \tag{2.1.2}$$

determines a closed embedding of the projectivization $\mathbb{X}$ of $\mathcal{I}_Z$ into $\mathbb{P}^r_X$. The ideal sheaf $\mathcal{I}_{\mathbb{X}}$ of $\mathbb{X}$ in $\mathbb{P}^r_X$ is locally generated by the entries of the row matrix $\begin{pmatrix} y_0 & \dots & y_r \end{pmatrix} p^* M$ where $y_0, \dots, y_r$ are the relative homogeneous coordinates of $\mathbb{P}^r_X$ and $p : \mathbb{P}^r_X \to X$ is the projection.

Remark 2.1.4. In the following, except in the examples, we consider locally free presentation

$$\mathcal{O}_X^m \otimes \mathcal{L}^\vee \xrightarrow{M} \mathcal{O}_X^{r+1} \otimes \mathcal{L}^\vee \xrightarrow{(\phi_0 \cdots \phi_r)} \mathcal{I} \longrightarrow 0 \tag{2.1.3}$$

of $\mathcal{I}$. In this case, the projectivization $\mathbb{P}(\mathcal{I})$ is embedded in $\mathbb{P}(\mathcal{O}_X^{r+1} \otimes \mathcal{L}^\vee)$. As we saw in Lemma 1.1.2, this is still considering $\mathbb{P}(\mathcal{I})$ in $\mathbb{P}^r_X$ but with a modified hyperplane class. This makes the redaction and verification of the proves easier.

We illustrate the previous notions with the following situation where the base field k is any field.

Let $\mathcal{I}_\mathcal{C} = (x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2)$ be the ideal sheaf over $\mathbb{P}^3$ given by the 2×2 minors of the matrix

$$M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

and denote those three minors by ϕ_0, ϕ_1, ϕ_2. As we explained in Remark 1.3.13, the subscheme $\mathcal{C} = \mathbb{V}(\mathcal{I}_\mathcal{C})$ of $\mathbb{P}^3$ is a smooth rational curve called the twisted cubic.

We summarize this situation into the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^2 \xrightarrow{{}^tM} \mathcal{O}^3_{\mathbb{P}^3} \xrightarrow{(\phi_0\ \phi_1\ \phi_2)} \mathcal{I}_\mathcal{C}(2) \longrightarrow 0 \tag{2.1.4}$$

Now, denoting by $\mathbb{X}$ the projectivization $\mathbb{P}\big(\mathcal{I}_\mathcal{C}(2)\big)$ of $\mathcal{I}_\mathcal{C}(2)$, (2.1.4) determines a closed embedding of $\mathbb{X}$ into $\mathbb{P}(\mathcal{O}^3_{\mathbb{P}^3})$. Moreover, denoting $\mathbb{P}(\mathcal{O}^3_{\mathbb{P}^3})$ by $\mathbb{P}^3 \times \mathbb{P}^2$ with variables x_i (resp. y_i) for the first (resp. second) factor, the ideal of $\mathcal{I}_\mathbb{X}$ of $\mathbb{X}$ in $\mathbb{P}^3 \times \mathbb{P}^2$ is generated by the entries in the row matrix

$$(y_0\ y_1\ y_2)^t M = \big(y_0x_0 + y_1x_1 + y_2x_2 \quad y_0x_1 + y_1x_2 + y_2x_3\big)$$

since by definition $\mathrm{Im}({}^tM)$ is the kernel of $\mathcal{O}^3_{\mathbb{P}^3} \to \mathcal{I}_\mathcal{C}(2) \to 0$.

By computing the primary decomposition of $\mathcal{I}_\mathbb{X}$, for example using basic functions of Macaulay2, we deduce that $\mathbb{X}$ has codimension 2 in $\mathbb{P}^3 \times \mathbb{P}^2$ and that it is irreducible. Now, we study the fibres of the morphism $\pi : \mathbb{X} \to \mathbb{P}^3$.

(i) If $z \notin \mathcal{C}$, then the localization $\mathcal{I}_{\mathcal{C},z}$ of $\mathcal{I}_\mathcal{C}$ at z is equal to $\mathcal{O}_z$. Since the formation of the symmetric algebra commutes with base change (Proposition 2.1.1), the fibre $\mathbb{X}_z$ is isomorphic to $\{z\}$. Actually, over the open subset $U = \mathbb{P}^3 \backslash \mathcal{C}$, we show by the same argument that $\mathbb{X}_U \simeq U$ since $\mathcal{I}_U \simeq \mathcal{O}_U$.

(ii) Now let $z = (1 : 0 : 0 : 0)$ so ${}^tM_z = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Hence, since the formation of the symmetric algebra commutes with base change and since $\mathbb{X}$ has equation $\begin{pmatrix} y_0 & y_1 & y_2 \end{pmatrix}^t M$, the fibre $\mathbb{X}_z$ of z in $\mathbb{X}$ is a line in $\mathbb{P}^2_z$.

As we are going to see in Subsection 2.1.2, $\mathbb{X}$ corresponds exactly here to the blow-up of $\mathbb{P}^3$ along $\mathcal{C}$.

Another perspective

We emphasize the following point. Since $\mathcal{I}_{\mathcal{C}}$ is generated by the 2×2-minors of the matrix $M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$, the structure sheaf $\mathcal{O}_{\mathcal{C}}$ of $\mathcal{C}$ fits also in the following exact sequence:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3) \xrightarrow{\begin{pmatrix} x_2^2 - x_1x_3 \\ -x_1x_2 + x_0x_3 \\ x_1^2 - x_0x_2 \end{pmatrix}} \mathcal{O}_{\mathbb{P}^3}(-1)^3 \xrightarrow{M} \mathcal{O}_{\mathbb{P}^3}^2 \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow 0$$

In this case, $\mathbb{D} = \mathbb{P}\big(\mathcal{O}_{\mathcal{C}}\big)$ is embedded in $\mathbb{P}(\mathcal{O}_{\mathbb{P}^3}^2) \simeq \mathbb{P}^3 \times \mathbb{P}^1$ and is a complete intersection of three divisors of class $c_1\big(\mathcal{O}_{\mathbb{P}^3\times\mathbb{P}^1}(0,1)\big)$ in the Picard group of $\mathbb{P}^3\times\mathbb{P}^1$.

Moreover, writing down explicitly those three divisors, we see that the fibre $\mathbb{D}_x$ of $\mathbb{D}$ over a point $x \in \mathbb{P}^3$ is defined by the ideal

$$(y_0x_0 + y_1x_1, y_0x_1 + y_1x_2, y_0x_2 + y_1x_3)$$

thus:

(1) if $x \in \mathcal{C}$ (i.e. $\operatorname{rank}(M_x) = 1$), $\dim(\mathbb{D}_x) = 0$,

(2) whereas if $x \notin \mathcal{C}$ (i.e. $\operatorname{rank}(M_x) = 2$), $\mathbb{D}_x$ is empty.

Hence $\pi(\mathbb{D})$ is set theoretically equal to $\mathcal{C}$. Actually, by definition of the structure sheaf of the projectivization $\mathbb{P}(\mathcal{O}_{\mathcal{C}})$ of $\mathcal{O}_{\mathcal{C}}$ we have also that $\pi_*(\mathcal{O}_{\mathbb{D}}) = \mathcal{O}_{\mathcal{C}}$.

2.1.2 Blow-up of an ideal sheaf

We turn now to the study of the Rees algebra of an ideal sheaf following [Har77, II.7]. Here, we explain how the data of the Rees algebra of the ideal sheaf of the base locus of a rational map Φ is equivalent to the data of the graph Γ of Φ.

For the rest of the subsection, X is a smooth quasi-projective variety.

Definition 2.1.5. Let $\mathcal{I}$ be a coherent ideal sheaf on X.

Consider the *Rees algebra*

$$\mathcal{R}(\mathcal{I}) = \oplus_{d\geq 0}\mathcal{I}^d t^d \subset \mathcal{O}_X[t]$$

where $\mathcal{I}^d$ is the dth power of the ideal $\mathcal{I}$, and where we set $\mathcal{I}^0 = \mathcal{O}_X$. Then $\mathcal{R}(\mathcal{I})$ satisfies ($\triangle$). We call $\mathbf{Proj}\big(\mathcal{R}(\mathcal{I})\big)$ the *blow-up of X with respect to the coherent sheaf of ideal $\mathcal{I}$* and we denote it by $\tilde{X}$.

If Z is the closed subscheme of X defined by $\mathcal{I}$, then we also call $\tilde{X}$ the *blow-up of X along Z*.

Example 2.1.6. Let X be the projective space $\mathbb{P}^3$ over any field k and let $\mathcal{I}$ be the ideal sheaf associated to the ideal (x_0, x_1, x_2). We denote also by z the point $\mathbb{V}(x_0, x_1, x_2)$ in $\mathbb{P}^3$. In this case the component $\mathcal{I}^d$ of degree d of $\mathcal{R}(\mathcal{I})$ is generated by all the monomials in x_0, x_1, x_2 of degree d. In other words $\mathcal{R}(\mathcal{I}) = \oplus_{d\geq 0}\mathcal{I}^d t^d$ is isomorphic to $\mathrm{k}[x_0, x_1, x_2]$ and the blow-up $\tilde{\mathbb{P}^3}$ with respect to $\mathcal{I}$ is equal to $\mathbb{X} = \mathbb{P}(\mathcal{I})$. So, as in Section 1.1, let:

$$\mathcal{O}^3_{\mathbb{P}^3}(-1) \xrightarrow{\begin{pmatrix} 0 & x_0 & -x_1 \\ -x_0 & 0 & x_2 \\ x_1 & -x_2 & 0 \end{pmatrix}} \mathcal{O}^3_{\mathbb{P}^3} \xrightarrow{\begin{pmatrix} x_0 & x_1 & x_2 \end{pmatrix}} \mathcal{I}(1) \longrightarrow 0$$

be a locally free presentation of $\mathcal{I}(1)$. By Lemma 1.1.2, $\mathbb{P}\big(\mathcal{I}(1)\big)$ is isomorphic to $\mathbb{X}$ so the ideal $\mathcal{I}_{\mathbb{X}}$ of $\mathbb{X}$ in $\mathbb{P}(\mathcal{O}^3_{\mathbb{P}^3}) \simeq \mathbb{P}^3 \times \mathbb{P}^2$ is equal to

$$(y_0x_1 - y_2x_0, -y_0x_2 + y_2x_0, y_1x_2 - y_2x_1).$$

We deduce from those equations, for example by computing its primary decomposition with Macaulay2, that $\tilde{\mathbb{P}^3}$ is irreducible of dimension 3. Furthermore, letting $U = \mathbb{P}^3 \backslash \{z\}$, we have that $\tilde{X}_U \simeq U$ and that $\sigma^{-1}(\{z\})$ is isomorphic to $\mathbb{P}^2_z \simeq \{z\} \times \mathbb{P}^2$.

The fact that $\mathcal{S} = \oplus_{d \geq 0} \mathcal{I}^d$ is a sheaf of integral domains on X explains Item (a) of the following proposition. We refer to [Har77, II.7.16] for the proof of Item (b) and Item (c).

Proposition 2.1.7. *Let $\mathcal{I} \subset \mathcal{O}_X$ be a non zero coherent sheaf of ideals on X, and let $\sigma : \tilde{X} \to X$ be the blowing-up with respect to $\mathcal{I}$.*

Then:

(a) *$\tilde{X}$ is a variety,*

(b) *σ is a birational, proper, surjective morphism,*

(c) *if X is quasi-projective over k then $\tilde{X}$ is also, and σ is a projective morphism.*

Now, let $\mathcal{I}$ be the base ideal sheaf of a rational map $\Phi = (\phi_0 : \ldots : \phi_n) : X \dashrightarrow \mathbb{P}^n$ associated to an $n+1$-subspace V of $\mathrm{H}^0(X, \mathcal{L})$ where $\mathcal{L}$ is a line bundle over X as in Section 1.3. We denote by Z the base scheme $\mathbb{V}(\mathcal{I})$ in X.

Proposition 2.1.8. *The blow-up $\sigma : \tilde{X} \to X$ along Z is isomorphic to the graph Γ of Φ embedded into $\mathbb{P}^n_X$.*

Proof. Let $U = W \backslash Z$. The representative $\langle \Phi_U, U \rangle$ of Φ to U is a morphism. Hence the graph $\Gamma_{\Phi_U} \simeq U$ of Φ_U is integral and its closure Γ in $\mathbb{P}^n_X$ is also integral.

Moreover Γ is a priori a subscheme of $\tilde{X}$. This follows from the universal property of $\tilde{X}$ [Har77, II.7.14]. Hence, since $\tilde{X}$ is also integral and coincide with Γ over U, by Proposition 2.1.7 (a), Γ and $\tilde{X}$ coincide. □

Example 2.1.9. Let X be a smooth quasi-projective variety over k and let $\mathcal{I} = (\phi_0, \ldots, \phi_n)$ be an ideal sheaf given by $n+1$ global sections of a line bundle $\mathcal{L}$ over X. We denote by $\Phi : X \dashrightarrow \mathbb{P}^n$ the associated map.

Proposition 2.1.8 gives a first approximation to compute the equations of the blow-up $\tilde{X}$ of X with respect to $\mathcal{I}$. Indeed, in $X \times \mathbb{P}^n$ with variables y in the second factor, let $\mathbb{K}$ be the scheme given by the 2×2 minors of the matrix $\begin{pmatrix} \phi_0 & \ldots & \phi_n \\ y_0 & \ldots & y_n \end{pmatrix}$. Since $\tilde{X}$ is isomorphic to the graph Γ of Φ by 2.1.8, we have that $\mathbb{K}$ contains $\tilde{X}$. As we will see, $\mathbb{K}$ contains also the projectivization $\mathbb{X}$ of $\mathcal{I}$.

2.2 Symmetric algebra versus Rees algebra

We explain now the relation between the blow-up of a given (coherent) ideal sheaf $\mathcal{I}$ on a smooth quasi-projective variety X and the projectivization $\mathbb{P}(\mathcal{I})$ i.e. the relation between the Rees algebra of $\mathcal{I}$ and the symmetric algebra of $\mathcal{I}$.

As in the previous section, X is a smooth quasi-projective variety, $\mathcal{I}$ is a coherent ideal sheaf over X and we denote by $\tilde{X}$ the blow-up of X along $\mathcal{I}$ and by $\mathbb{X}$ the projectivization of $\mathcal{I}$.

Proposition 2.2.1. *The blow-up $\tilde{X}$ is an irreducible component of $\mathbb{X}$.*

Proof. Denoting $\mathcal{T}(\mathcal{I})$ the tensor algebra associated to $\mathcal{I}$ and $\mathcal{T}^n(\mathcal{I})$ its component of degree n, let $\mathrm{Q}^n : \mathcal{T}^n(\mathcal{I}) \to \mathcal{I}^n t^n$ be the surjection sending a pure tensor $x_1 \otimes \ldots \otimes x_n \in \mathcal{T}^n(\mathcal{I})$ to $x_1 \ldots x_n t^n$. It induces a surjection $\mathrm{Q} : \mathcal{T}(\mathcal{I}) \to \mathcal{R}(\mathcal{I})$ whose kernel contains the symmetric relations, i.e. the subsheaf generated by all the sums $x \otimes y - y \otimes x$. Hence Q induces a surjection $\mathrm{R} : \mathrm{Sym}(\mathcal{I}) \to \mathcal{R}(\mathcal{I})$. This shows the injection $\tilde{X} \hookrightarrow \mathbb{X}$ at the **Proj** level. Since $\tilde{X}$ is integral by 2.1.7 (a) and coincide with $\mathbb{X}$ over the complement of $\mathbb{V}(\mathcal{I})$ in X, $\tilde{X}$ is an irreducible component of $\mathbb{X}$. □

Of course, the blow-up $\tilde{X}$ and the projectivization $\mathbb{X}$ are equal when the surjection $\mathrm{R} : \mathrm{Sym}(\mathcal{I}) \to \mathcal{R}(\mathcal{I})$ is injective and we explain now a sufficient condition when it is the case.

2.2.1 Local complete intersections

We define first *regular sequences* in the local case. Recall that given a module M over a ring R, an element $r \in R$ is called a *nonzerodivisor* if $rm \neq 0$ for all $m \in M \backslash \{0\}$.

Definition 2.2.2. Let R be a noetherian ring and M an R-module. A sequence of elements $r_0, \ldots, r_n \in R$ is called *M-regular* if:

(1) $(r_0, \ldots, r_n)M \neq M$,

(2) for $i = 1, \ldots, n$, r_i is a nonzerodivisor on $M/(r_1, \ldots, r_{i-1})M$.

When $M = R$, a R-regular sequence is simply called a *regular sequence*.

Example 2.2.3. With $R = \mathrm{k}[x_0, x_1, x_2]$, the sequence $\big(x_0, (1-x_0)x_1, (1-x_0)x_2\big)$ is regular whereas $\big((1-x_0)x_1, (1-x_0)x_2, x_0\big)$ is not.

Definition 2.2.4. Let R be a noetherian ring, M an R-module and I an ideal of R. If $IM \neq M$, then by [Eis95, Theorem 17.4], the length of all maximal M-regular sequences in I are the same. We define the *depth* of I on M, denoted by $\mathrm{depth}(I, M)$, to be the length of any maximal M-regular sequence in I. If $M = R$, we call just the *depth* of I, written $\mathrm{depth}(I)$.

One interest in regular sequences $(r_0, \ldots, r_n)$ lies in the remarkable structure of the resolution of the module $M = R/I$ where I is the ideal generated by $r_0, \ldots, r_n$. Namely the *Koszul complex* resolves I. We introduce the Koszul complex following [Har77, III.7].

Definition 2.2.5. Let R be a ring and $r_0, \ldots, r_n \in R$. We define the *Koszul complex* $K_\bullet(r_0, \ldots, r_n)$ as follows: K_1 is a free R-module of rank $n+1$ with basis $e_0, \ldots, e_n$. For each $p = 0, \ldots, n$, $K_p = \wedge^p K_1$. We define the boundary map $d : K_p \to K_{p-1}$ on the basis vectors:

$$d(e_{i_1} \wedge \ldots \wedge e_{i_p}) = \sum_{j=1}^{p} (-1)^{j-1} r_{i_j} e_{i_1} \wedge \ldots \wedge \hat{e_{i_j}} \wedge \ldots \wedge e_{i_p}$$

which verifies that $d_p \circ d_{p-1} = 0$ i.e. $K_\bullet(r_0, \ldots, r_n)$ is a complex of R-module. If M is any R-modules, we set $K_\bullet(r_0, \ldots, r_n, M) = K_\bullet(r_0, \ldots, r_n) \otimes M$.

The homology of the complex of R-modules $K_\bullet(r_0, \ldots, r_n, M)$ is called the *Koszul homology* associated to the Koszul complex.

Proposition 2.2.6. *[Har77, Proposition III.7.10A] Let R be a noetherian ring, M an R-module and $r_0, \ldots, r_n$ a M-regular sequence, then:*

$$H_i\big(K_\bullet(r_0, \ldots, r_n, M)\big) = 0 \quad \textit{for } i > 0$$

and

$$H_0\big(K_\bullet(r_0, \ldots, r_n, M)\big) = M/(r_0, \ldots, r_n)M$$

where $H_i\big(K_\bullet(r_0, \ldots, r_n, M)\big)$ is the ith homology module of the complex of R-modules $K_\bullet(r_0, \ldots, r_n, M)$.

Example 2.2.7. Let $R = \mathrm{k}[x, y, z]$ and I be the ideal (x, y, z). The sequence (x, y, z) is regular so the Koszul complex $K_\bullet(x, y, z, R)$ is a free resolution of R/I. Namely:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} I \longrightarrow 0$$

is an exact complex.

We explain now how the notion of regular sequence is relevant when considering Rees and symmetric algebra.

Proposition 2.2.8. *If $I = (x_0, \ldots, x_n) \subset R$ is generated by a regular sequence, then*

$$\mathcal{R}(I) \simeq \mathrm{Sym}(I) \simeq R[y_0, \ldots, y_n]/J$$

where J is generated by the 2×2 minors of the matrix

$$\begin{pmatrix} x_0 & \ldots & x_n \\ y_0 & \ldots & y_n \end{pmatrix}.$$

Proof. Since $(x_0, \ldots, x_n)$ is regular, the Koszul complex $K_\bullet(x_0, \ldots, x_n, R)$ resolves R/I so I has the following presentation:

$$R^{\binom{n+1}{2}} \xrightarrow{\begin{pmatrix} -x_1 & x_2 & \cdots & 0 \\ x_0 & 0 & \cdots & \vdots \\ 0 & -x_0 & \cdots & \vdots \\ \vdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & x_n \\ 0 & 0 & \cdots & -x_{n-1} \end{pmatrix}} R^{n+1} \xrightarrow{\begin{pmatrix} x_0 & \dots & x_n \end{pmatrix}} I \longrightarrow 0$$

where the presentation matrix is the second differential D_2 of the Koszul complex. Actually, the ideal generated by the row matrix $(y_0 \ \dots \ y_n)D_2$ is the same as the one generated by the 2×2 minors of the matrix $\begin{pmatrix} x_0 & \dots & x_n \\ y_0 & \dots & y_n \end{pmatrix}$, D_2 being the presentation matrix of the ideal $(x_0, \dots, x_n)$. Hence $\mathrm{Sym}(I) \simeq R[y_0, \dots, y_n]/J$. The proposition follows from the isomorphism $\mathrm{Sym}(I) \simeq \mathcal{R}(I)$ implied by the fact that a permutation in a regular sequence is a regular sequence (see [Eis95, 17.2] and [EH00, Exercise IV-26] for a complete treatment of this last argument). □

Example 2.2.9. In the case of an ideal $I = (x, y)$ generated by a regular sequence of length 2, we thus have

$$\mathcal{R}(I) \simeq \mathrm{Sym}(I) \simeq R[u, v]/(uy - vx).$$

As an illustration of Proposition 2.2.8, assume that $I = (x_0, \dots, x_n)$ is generated by $n + 1$ elements of R and let

$$R^m \xrightarrow{M} R^{n+1} \longrightarrow I \longrightarrow 0$$

be a presentation of I. The generators of the symmetric algebra $\mathrm{Sym}(I)$ of $\mathcal{I}$ viewed as a quotient of $R[y_0, \dots, y_n]$ are the entries in the row matrix $(y_0 \ \dots \ y_n)M$ as we explained in Section 1.1. Hence if $\mathcal{R}(I) = \mathrm{Sym}(I)$, the generators of $\mathcal{R}(I)$ are linear in the y variables. This motivates the following definition.

Definition 2.2.10. [Vas05] Given a noetherian ring R and an ideal I over R, I is said of *linear type* if $\mathcal{R}(I) = \mathrm{Sym}(I)$.

2.2.2 Torsion of the symmetric algebra

Now, we determine the kernel of the surjection $\mathrm{Sym}(\mathcal{I}) \to \mathcal{R}(\mathcal{I})$.

Lemma 2.2.11. *[Mic64] Let X be a smooth quasi-projective variety and let $\mathcal{I}$ be an ideal sheaf over $\mathcal{O}_X$.*

Then

$$\mathcal{R}(\mathcal{I}) = \mathrm{Sym}(\mathcal{I})/\mathrm{T}(\mathcal{I})$$

where $\mathrm{T}(\mathcal{I})$ is the torsion sheaf $\{u \in \mathrm{Sym}(\mathcal{I}),\ \exists r \in \mathcal{O}_X,\ ru = 0\}$ of the symmetric algebra.

Proof. Since the formation of the symmetric algebra and the Rees algebra commutes with base change (see 2.1.1), we can assume that X is an affine scheme $\mathrm{Spec}(A)$ with A a local ring. It is thus enough to show Lemma 2.2.11 for an ideal I generated by $\phi_0, \dots, \phi_n$ over an integral noetherian ring A. As we explained in

Subsection 2.1.1, by [Bou70, A.III.69.4], $\mathrm{Sym}(I)$ is isomorphic to $A[y_0, \ldots, y_m]/Q$ where Q is the A-ideal generated by the linear polynomials $b_0y_0 + \ldots + b_my_m$ such that $b_0\phi_0 + \ldots + b_m\phi_m = 0$.

Consider the surjection

$$\begin{aligned} g: \quad A[y_0, \ldots, y_m] &\longrightarrow \mathcal{R}(I) = \oplus_{j\geq 0} I^j t^j. \\ y_i &\longmapsto \phi_i t \end{aligned}$$

First, $Q \subset Q_\infty = \ker(g)$ and, since A is integral, the torsion module $\mathrm{T}(\mathcal{I})$ of $\mathrm{Sym}(\mathcal{I})$ is also contained in Q_∞/Q. Moreover, letting $F \in A[y_0, \ldots, y_m]$ and $F = F_0 + F_1 + \ldots$ its decomposition into homogeneous polynomials, we have that $F \in Q_\infty$ if and only if $F_i \in Q_\infty$ for any i. Moreover, any homogeneous polynomial F is in Q_∞ if and only if $F(\phi_0, \ldots, \phi_n) = 0$. Hence $\ker\big(\mathrm{Sym}(I) \to \mathcal{R}(I)\big) = Q_\infty/Q$ which contains $\mathrm{T}(I)$. Now we show the reverse inclusion $Q_\infty/Q \subset \mathrm{T}(I)$, that is, for all element $f \in Q_\infty$ there exist $r \in A$ such that $rf \in Q$. We proceed by induction on the degree δ of f.

If $\delta = 1$, we have $Q_\infty = Q$ by definition of Q so given any $f \in Q_\infty$ of degree 1, $1 \times f \in Q$ which shows the initialisation of the induction. Now, for any $\delta > 1$, suppose the result true for any P of degree $\delta - 1$ and take $f \in \mathrm{T}(I)$. Write

$$f = y_0f_0(y_0, \ldots, y_n) + y_1f_1(y_1, \ldots, y_n) + \ldots + y_nf_n(y_n)$$

and write $h = y_0f_0(\phi_0, \ldots, \phi_n) + y_1f_1(\phi_1, \ldots, \phi_n) + \ldots + y_nf_n(\phi_n)$. Since h has degree 1 in Q_∞ it is in Q. Now write,

$$\phi_n^{\delta-1}f - y_n^{\delta-1}h = y_0h_0(y_0, \ldots, y_n) + \ldots + y_{n-1}h_{n-1}(y_{n-1}, y_n)$$

for h_i homogeneous of degree $\delta - 1$ in Q_∞. By the induction hypothesis, there are $r_i \in R$ such that $r_ig_i \in Q$. Then $r = r_0 \ldots r_{n-1}\phi_n^{\delta-1} \in R$ is an element such that $rf \in Q$. □

Geometric description of the torsion

We turn to the geometric point of view of Lemma 2.2.11. Let X be a smooth quasi-projective variety and let $\mathcal{I}$ be an ideal sheaf on X. We denote by $\mathbb{X}$ the projectivization of $\mathcal{I}$ with its structure map $\pi_1 : \mathbb{X} \to X$ and by $\tilde{X}$ the blow-up of X with respect to $\mathcal{I}$ and with structure map $\sigma_1 : \tilde{X} \to X$. The images by π_1 of the irreducible components of $\mathbb{X}$ different from $\tilde{X}$ are contained in the support of Z. Indeed, over the set $U = X\backslash Z$, we have $\mathcal{I}_U = \mathcal{O}_U$, so that $\tilde{X}|_U = \mathbb{X}|_U = \pi_1^{-1}(U)$. This justifies the following definition:

Definition 2.2.12. An irreducible component of the projectivization $\mathbb{X}$ of $\mathcal{I}$ different from $\tilde{X}$ is called a *torsion component* of X. The union of the torsion components is called the *torsion part* of $\mathbb{X}$, denoted by $\mathbb{T}_Z$.

The following proposition provides a way to detect the torsion components of $\mathbb{X}$ and to describe them with algebraic properties of Z.

Proposition 2.2.13. *Let $x \in X$ be a closed point and let*

$$\begin{tikzcd} \mathcal{P}_2 \arrow[rr, "M"] \arrow[dr] & & \mathcal{P}_1 \arrow[r, "\Phi"] & \mathcal{I}_Z \arrow[r] & 0 \\ & \mathcal{E} \arrow[ur] \arrow[dr] & & & \\ 0 \arrow[ur] & & 0 & & \end{tikzcd} \tag{2.2.1}$$

be a locally free presentation of $\mathcal{I}_Z$ where $\mathcal{P}_1$ has rank $n+1$ and where $\mathcal{E}$ is the image of M. The scheme-theoretic fibre $\mathbb{X}_x$ when equipped with its reduced structure is isomorphic to $\mathbb{P}^{n-r}_x$ where $r = \operatorname{rank}(M_x)$.

Proof. Since the formation of the symmetric algebra commutes with base change (see 2.1.1), the fibre $\mathbb{X}_x$ is obtained by localizing X at x and taking $\mathbb{P}(\mathcal{I}_Z \otimes \mathrm{k}_x)$, where k_x is the residue field of $\mathcal{O}_X$ at x. Now tensor (2.2.1) by k_x and observe that the kernel $\mathcal{K}_x$ of the surjection $\Phi_x : \mathcal{P}_1 \otimes \mathrm{k}_x \to \mathcal{I}_Z \otimes \mathrm{k}_x$ is a quotient of $\mathcal{E} \otimes \mathrm{k}_x$, which in turn is a quotient of $\mathcal{P}_2 \otimes \mathrm{k}_x$. The composition of these surjections and of the inclusion $\mathcal{K}_x \to \mathrm{V}$ is just the matrix M_x, so $\ker(\Phi_x) = \operatorname{Im}(M_x)$. Therefore $\dim(\mathcal{I}_Z \otimes \mathrm{k}_x) = n + 1 - \operatorname{rank}(M_x)$, which completes the proof. □

From a practical point of view, it might be difficult to determine how the fibres (hence, as we will see, the torsion components) vary from a given presentation, as illustrated by the following example:

Example 2.2.14. Consider the ideal I of $A = \mathrm{k}[x, y, z]$ generated by the 3×3-minors $\phi_0, \ldots, \phi_3$ of the matrix

$$\begin{pmatrix} 0 & xz & y^2 \\ 0 & x & xy \\ x & y & y \\ y & z & x \end{pmatrix}.$$

Since $\mathbb{V}(I)$ has the expected codimension 2 in $\operatorname{Spec}(A)$, we can apply the Hilbert-Burch theorem (see [Eis95, 20.15] or Proposition 1.4.9 for this theorem) to show that a minimal free resolution reads:

$$0 \longrightarrow A^3 \xrightarrow{\begin{pmatrix} 0 & xz & y^2 \\ 0 & x & xy \\ x & y & y \\ y & z & x \end{pmatrix}} A^4 \xrightarrow{(\phi_0 \ \ldots \ \phi_3)} I \longrightarrow 0$$

Hence, above the line $\{x = y = 0\}$ in $X = \operatorname{Spec}(A)$, the fibre is $\{x = y = 0\} \times \mathbb{P}^2_{\mathrm{k}}$ but above the point $\{x = y = z = 0\}$, the fibre is $\{x = y = z = 0\} \times \mathbb{P}^3_{\mathrm{k}}$. Actually, after computing the primary decomposition of the projectivization $\mathbb{X}$ of I with, for instance, Macaulay2, these fibres are the torsion components of $\mathbb{X}$.

Determinants and Fitting ideals

As we saw in the proof of Proposition 2.2.13, the location of the torsion components is computed by the rank of the presentation matrix M motivating the following definition.

Definition 2.2.15. Let X be an algebraic variety and $\mathcal{F}$, $\mathcal{G}$ be vector bundles on X, of ranks f, g respectively. For any vector bundle map $s : \mathcal{F} \to \mathcal{G}$ and $l \in \mathbb{N}$ verifying $0 \leq l \leq \min\{f, g\}$, the l'th degeneracy locus of s is the set

$$X_l(s) = \{x \in X | \operatorname{rank} s(x) \leq l\}.$$

Since a linear map has rank $\leq l$ if and only if all $l+1$-minors vanish, the set $X_l(s)$ can also be described as the zero locus of the section $\wedge^{l+1} s \in \mathrm{H}^0(X, (\wedge^{k+1}\mathcal{F})^\vee \otimes \wedge^{k+1}\mathcal{G})$. As such, it comes with a natural structure as a closed subscheme of X. We denote by $\mathcal{I}_l(s)$ its associated ideal sheaf and we call it *the ideal sheaf of minors of size l of s* by extension.

Proposition 2.2.16. *[Eis95, 20.4] Let $\mathcal{I}$ be an ideal sheaf of the structure sheaf $\mathcal{O}_X$ of a quasi-projective variety X and let $\mathcal{F} \xrightarrow{s} \mathcal{O}_X^{n+1} \to \mathcal{I} \to 0$ and $\mathcal{F}' \xrightarrow{s'} \mathcal{O}_X^{n'+1} \to \mathcal{I} \to 0$ be two presentations of $\mathcal{I}$ where $\mathcal{F}$ (resp. $\mathcal{F}'$) is locally free of rank r (resp. rank r').*

Then, for each integer $i \geq 0$, we have that $\mathcal{I}_{n-i}(s) = \mathcal{I}_{n'-i}(s')$.

This justifies the following definition.

Definition 2.2.17. We define the ith Fitting ideal of $\mathcal{I}$ to be the ideal sheaf

$$\operatorname{Fitt}_i(\mathcal{I}) = \mathcal{I}_{n-i}(s)$$

for a given a presentation morphism s of $\mathcal{I}$ whose image has rank r.

We describe in more details the special case when $\mathcal{I}$ is generated by $n+1$ global sections of a line bundle $\mathcal{L}$, n being the dimension of X, and $\mathbb{V}(\mathcal{I})$ is zero-dimensional in X. Let

$$\mathcal{F} \xrightarrow{s} \mathcal{O}_X^{n+1} \xrightarrow{(\phi_0 \ \dots \ \phi_n)} \mathcal{I} \otimes \mathcal{L} \longrightarrow 0$$

be a locally free presentation of $\mathcal{I} \otimes \mathcal{L}$. The map s can be interpreted dually as the data of $n+1$ sections of $\mathcal{F}^\vee$.

Proposition 2.2.18. *Under the previous settings, the ideal sheaf $\operatorname{Fitt}_{n-1}\mathcal{I}$ is the ideal sheaf generated by the common vanishing of these $n+1$ sections and $\mathbb{V}(\operatorname{Fitt}_{n-1}\mathcal{I}) \subset Z = \mathbb{V}(\mathcal{I})$.*

Proof. Since $\operatorname{rank}(\mathcal{F}) = n$, $\operatorname{Fitt}_{n-1}\mathcal{I}$ is equal to the ideal sheaf $\mathcal{I}_1(M)$ of the minors 1×1 of M which means that it is generated by the common vanishing of the $n+1$ sections of $\mathcal{F}^\vee$. The fact that $\mathbb{V}(\operatorname{Fitt}_{n-1}\mathcal{I}) \subset Z = \mathbb{V}(\mathcal{I})$ follows from [Eis95, 20]. □

Proposition 2.2.19. *Let X be a smooth quasi-projective variety of dimension n over k and let $\mathcal{I}$ be an ideal sheaf over X. Denoting $Z = \mathbb{V}(\mathcal{I})$, assume that $\operatorname{codim}(Z) = n$ and that $\mathcal{I} \otimes \mathcal{L}$ is generated by $n+1$ sections for some line bundle $\mathcal{L}$ over X. Then the images of the torsion components of $\mathbb{X}$ in X are precisely supported on the points of the subscheme $\mathbb{V}(\operatorname{Fitt}_{n-1}\mathcal{I})$. Moreover, each torsion component is isomorphic to $\mathbb{P}^n_{\mathrm{k}}$ when equipped with its reduced structure.*

Proof. Since Z is zero-dimensional, any $z \in Z$ is in an affine open set $U = \mathrm{Spec}(A)$ of X over which $\mathcal{L}$ is trivial. So over U, let

$$\mathcal{O}_U^m \xrightarrow{M} \mathcal{O}_U^{n+1} \xrightarrow{\begin{pmatrix}\phi_0 & \dots & \phi_n\end{pmatrix}} \mathcal{I}_{|U} \longrightarrow 0$$

be a presentation of $\mathcal{I}$.

Let z be a (closed) point of $\mathbb{V}\big(\mathcal{I}_1(M)\big)$ i.e. assume that M vanishes at z. Then the fibre of the projection $p : \mathbb{X} \to X$ is $\mathbb{P}^n_z$. This fibre is thus contained in the reduced structure of an irreducible component of $\mathbb{X}$ which is distinct from the blow-up $\tilde{X}$ of X with respect to $\mathcal{I}$. It is therefore contained in a torsion component. This in turn is clear as the restriction of p to $\tilde{X}$ is birational and the $p_{|\tilde{X}}$-fibre at any point has dimension at most $n-1$. Hence, we have proved that any point in $\mathbb{V}\big(\mathcal{I}_1(M)\big)$ lies in the image of a torsion component.

Conversely, take a (closed) point $x \in \mathbb{X}$, put $z = p(x)$ and assume that z lies away from $\mathbb{V}\big(\mathcal{I}_1(M)\big)$. First, if $z \notin Z$ then x lies in no torsion component, as all such components map to the base locus Z. Then, if $z \in Z$, since the formation of the symmetric algebra commutes with base change (2.1.1), up to localizing X at z we may assume that $X = \mathrm{Spec}(A)$ with A regular local ring. Then, since $z \notin \mathbb{V}\big(\mathcal{I}_1(M)\big)$, by [Eis95, Proposition 20.6] we argue that $\mathcal{I}$ can be generated (locally around z) by a regular sequence $(\psi_1, \dots, \psi_n)$. Therefore, by Proposition 2.2.8 we have that $\mathcal{R}(\mathcal{I}) = \mathrm{Sym}(\mathcal{I})$ so that there is no torsion component in $\mathbb{X}$. Clearly, this implies that x does not belong to any torsion component.

Hence, the torsion part of $\mathbb{X}$ is supported over $\mathbb{P}^n_{\mathbb{V}(I_1(M))}$ whose irreducible components (i.e. the torsion components of $\mathbb{X}$) are supported over $\mathbb{P}^n_z$ for $z \in \mathbb{V}(I_1(M))$ since Z is 0-dimensional. □

Notation 2.2.20. In the situation of Proposition 2.2.19, for every $z \in Z$, we let $\mathbb{T}_z$ be the scheme-theoretic fibre of the restriction of π_1 to $\mathbb{T}_Z$. By Proposition 2.2.19, $\mathbb{T}_z$ is set-theoretically equal to $\mathbb{P}^n_z$ so $T_z = [\mathbb{T}_z] \cdot c_1\big(\mathcal{O}_{\mathbb{X}}(1)\big)^n$ is a 0-cycle on $\mathbb{X}$. We denote by T_Z the 0-cycle $\sum_{z \in Z} T_z$.

Generalised Milnor and Tjurina numbers

Anticipating on Subsection 6.2.1 about the possible measures of the difference between Rees and symmetric algebra, let us illustrate here a way to generalise Milnor and Tjurina numbers (see Definition 8 and Definition 16 for their usual definitions). So let X be a smooth quasi-projective variety of dimension n, $\mathcal{I}$ be an ideal sheaf generated by $n+1$ global sections of a line bundle $\mathcal{L}$ over X and assume that $Z = \mathbb{V}(\mathcal{I})$ is zero-dimensional.

Definition 2.2.21. With notation as in Notation 2.2.20, for every $z \in Z$, put:

- $\tau(Z, z) = \mathrm{length}(\mathcal{O}_{Z,z})$
- $\mu(Z, z) = \tau(Z, z) + \deg(T_z)$.

We let $\tau(Z) = \sum_{z \in Z} \tau(Z, z)$ and $\mu(Z) = \sum_{z \in Z} \mu(Z, z)$.

The result is that, when $X = \mathbb{P}^n$, and the $n+1$ sections are the partial derivatives of a square free homogeneous polynomial $f \in \mathrm{k}[x_0, \ldots, x_n]$, the numbers $\mu(Z)$ and $\tau(Z)$ coincide with the usual Milnor and Tjurina numbers $\mu_f(Z)$ and $\tau_f(Z)$ defined in Definition 8 and Definition 16.

Proposition 2.2.22. *Let $F = \{f = 0\}$ be a reduced hypersurface in $\mathbb{P}^n$ where f is a homogeneous polynomial in $\mathrm{k}[x_0, \cdots, x_n]$ of degree d and let $\mathcal{I}$ be the ideal sheaf generated by the partial derivatives of f. Let $z \in Z = \mathbb{V}(\mathcal{I})$ then:*

$$\tau(Z, z) = \tau_f(Z, z) \quad \text{and} \quad \mu(Z, z) = \mu_f(Z, z).$$

We postpone the proof of this result to Subsection 6.2.1.

Chapter 3

Resolution of the symmetric algebra

Given a smooth quasi-projective variety X of dimension n and an ideal sheaf $\mathcal{I}$ generated by $n+1$ global sections of a line bundle $\mathcal{L}$ over X, we saw in Chapter 2 that the equations of the projectivization $\mathbb{X}$ of $\mathcal{I}$ in $\mathbb{P}^n_X$ are determined by a locally free presentation of $\mathcal{I}$. Namely, letting

$$\mathcal{F} \xrightarrow{M} \mathcal{O}_X^{n+1} \to \mathcal{I} \otimes \mathcal{L} \to \ 0$$

be a locally free presentation of $\mathcal{I} \otimes \mathcal{L}$, the equations of $\mathbb{X}$ in $\mathbb{P}^n_X$ are the entries of the row matrix $\begin{pmatrix} y_0 & \dots & y_n \end{pmatrix} M$, where y_i are the relative homogeneous coordinates of $\mathbb{P}^n_X$. If $\mathcal{F}$ has rank n then $\mathbb{X}$ is the intersection of n divisor. Hence if $\mathbb{X}$ has dimension n the collection of entries in $\begin{pmatrix} y_0 & \dots & y_n \end{pmatrix} M$ is a regular sequence and the resolution of the ideal $\mathcal{I}_{\mathbb{X}}$ of $\mathbb{X}$ in $\mathbb{P}^n_X$ is thus the Koszul complex associated the sequence $\begin{pmatrix} y_0 & \dots & y_n \end{pmatrix} M$, see Definition 2.2.5 for the definition of the Koszul complex. This fact is of great interest for us since it implies the vanishing of some higher direct image sheaves when pushing-forward the Koszul complex by the projection map, as we will see. However, in greater generality, i.e. when $\operatorname{rank}(\mathcal{F}) > n$, the Koszul complex does not resolve $\mathcal{I}_{\mathbb{X}}$ anymore. Our aim in this chapter is to prove that, provided that $\mathbb{V}(\mathcal{I})$ is zero-dimensional, the resolution of $\mathcal{I}_{\mathbb{X}}$ keeps a remarkable property which we call subregularity, see Definition 3.2.3. This property insures that the same type of vanishing of higher direct image sheaves holds as in the "Koszul case".

Let us present more precisely our motivation with the following example. We refer to Section 1.2 and Subsection 1.3.3 for the associated definitions of Chow ring and to Subsection 3.1.1, Subsection 3.1.2 and Subsection 3.1.3 for the definitions of push-forward and Eagon-Northcott complex. This example is also a presentation of the problem in Chapter 6.

Example 3.0.1. Let $\mathcal{I}$ be the ideal sheaf over $\mathcal{O}_{\mathbb{P}^2}$ generated by the 2×2-minors of the matrix $M = \begin{pmatrix} x_0 & x_0^2 \\ x_1 & x_1^2 \\ x_2 & x_2^2 \end{pmatrix}$ so that a locally free resolution of $\mathcal{I}$ reads:

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-1) \oplus \mathcal{O}_{\mathbb{P}^2}(-2) \to \mathcal{O}^3_{\mathbb{P}^2} \to \mathcal{I}(3) \to 0$$

and a locally free resolution of the projectivization $\mathbb{X}$ of $\mathcal{I}$ reads

$$0 \to \mathcal{O}_{\mathbb{P}}(-3,-2) \to \mathcal{O}_{\mathbb{P}}(-1,-1) \oplus \mathcal{O}_{\mathbb{P}}(-2,-1) \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{X}} \to 0,$$

where $\mathbb{P} = \mathbb{P}^2 \times \mathbb{P}^2 = \mathbb{P}(\mathcal{O}_{\mathbb{P}^2}[y_0, \ldots, y_n])$ and where we write to the left the shift in variables $x_0, \ldots, x_n$ and right the shift in the variables $y_0, \ldots, y_n$. The naive projective degree $\mathfrak{d}_2$ of Φ is by definition the coefficient of h_1^2 in the decomposition of $\mathbb{X}$ in the Chow ring of $\mathbb{P}$ and where h_1 is the pull back of the hyperplane class of the first factor of $\mathbb{P}$ (see Definition 4.1.1 for the definition of naive projective degrees). In this case, the decomposition of $\mathbb{X}$ reads:

$$[\mathbb{X}] = (h_1 + h_2)(2h_1 + h_2) = 2h_1^2 + 3h_1h_2 + h_2^2$$

since $\mathbb{X}$ is complete intersection and we have that $\mathfrak{d}_2 = 2$.

Equivalently, the naive projective degree $\mathfrak{d}_2$ is the length of the scheme W defined as the support of the cokernel of a general map $\mathcal{O}^2_{\mathbb{P}} \to \mathcal{O}_{\mathbb{X}}(0,1)$ i.e. we have the following exact sequence:

$$\mathcal{O}^2_{\mathbb{P}} \to \mathcal{O}_{\mathbb{X}}(0,1) \to \mathcal{O}_W(0,1) \to 0. \tag{3.0.1}$$

Let $p_1 : \mathbb{P} \to \mathbb{P}^2$ be the first projection. As we will explain in Subsection 3.1.1, we can push forward by p_1 the sequence (3.0.1) to obtain the exact sequence

$$p_{1*}\mathcal{O}^2_{\mathbb{P}} \to p_{1*}\mathcal{O}_{\mathbb{X}}(0,1) \to p_{1*}\mathcal{O}_W(0,1). \tag{3.0.2}$$

In this sequence, we have that $p_{1*}\mathcal{O}^2_{\mathbb{P}} \simeq \mathcal{O}^2_{\mathbb{P}^2}$. Now let tentatively assume that $p_{1*}\mathcal{O}_{\mathbb{X}}(0,1) \simeq \mathcal{I}(3)$ and that the last map in (3.0.2) is surjective. As we will see this is not an obvious fact. Then we have the following exact sequence

$$\mathcal{O}^2_{\mathbb{P}^2} \to \mathcal{I}(3) \to p_{1*}\mathcal{O}_W(0,1) \to 0$$

The length of $p_{1*}\mathcal{O}_W(0,1)$ is actually the length of a general cosection of the image sheaf $\mathcal{E}$ of the presentation matrix M. So provided that $p_{1*}\mathcal{O}_{\mathbb{X}}(0,1) \simeq \mathcal{I}(3)$ and that the last morphism in (3.0.2) is surjective, we can compute the 2nd naive topological degree on X. As we will explain, the two assumptions are verified since, under our hypothesis, the resolution of $\mathbb{X}$ over $\mathcal{O}_{\mathbb{P}}$ is subregular (the verification of these two assumptions is precisely the problem of Chapter 6).

3.1 Algebraico-geometric background

3.1.1 Cohomological results about direct image sheaves

For this background about direct image sheaves, in particular about right derived functors, we refer to [Har77, III.1 Derived functors].

Definition 3.1.1. Let $p : X \to Y$ be a continuous map of topological spaces. For any sheaf of abelian groups $\mathcal{F}$ on X, the *direct image* sheaf $p_*\mathcal{F}$ on Y is the sheaf defined by $p_*\mathcal{F}(V) = \mathcal{F}\big(f^{-1}(V)\big)$ for any open set $V \subset Y$.

Note that p_* is a functor from the category $\mathfrak{Ab}(X)$ of sheaves of abelian groups on X to the category $\mathfrak{Ab}(Y)$ of sheaves on Y. Actually, p_* is left exact meaning that p_* sends any exact sequence:

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0 \tag{3.1.1}$$

to the exact sequence

$$0 \to p_*\mathcal{F}' \to p_*\mathcal{F} \to p_*\mathcal{F}''.$$

Moreover, $\mathfrak{Ab}(X)$ has enough injectives, see [Har77, III.1]) so we can state the following definition:

Definition 3.1.2. Let $p : X \to Y$ be a continuous map of topological spaces. Then the *higher direct image* functors $R^ip_* : \mathfrak{Ab}(X) \to \mathfrak{Ab}(Y)$ is the right derived functors of p_*.

For us the main use of the derived theory is as follows. The push forward of (3.1.1) gives by definition the long exact sequence:

$$\begin{aligned} 0 \longrightarrow p_*\mathcal{F}' \longrightarrow p_*\mathcal{F} \longrightarrow p_*\mathcal{F}'' \longrightarrow R^1p_*\mathcal{F}' \to R^1p_*\mathcal{F} \to \\ R^1p_*\mathcal{F}' \to \cdots \to R^ip_*\mathcal{F}' \to R^ip_*\mathcal{F} \to R^ip_*\mathcal{F}' \longrightarrow \cdots \end{aligned}$$

where each of the sheaves involved can be computed thanks to the cohomology of $\mathcal{F}$, $\mathcal{F}'$ and $\mathcal{F}''$. More precisely:

Proposition 3.1.3. *[Har77, III.8.5] Let X be a noetherian scheme, and let $p : X \to Y$ be a morphism from X to an affine scheme $Y = \mathrm{Spec}(A)$. Then for any quasi-coherent sheaf $\mathcal{F}$ on X, we have*

$$\mathrm{R}^ip_*(\mathcal{F}) \simeq \mathrm{H}^i(X, \mathcal{F})^{\sim}.$$

The gain is that, in our context, we can compute the cohomology of sheaves thank to the two following theorems.

Theorem 3.1.4. *[Har77, III.2.7] Let X be a noetherian topological space of dimension n. Then for all $i > n$ and all sheaves of abelian groups $\mathcal{F}$ on X, we have $\mathrm{H}^i(X, \mathcal{F}) = 0$.*

Theorem 3.1.5. *[Har77, III.5.1] Let A be a noetherian ring, and let $X = \mathbb{P}^n_A$ with $n \geq 1$ be the projective space of dimension n over A. Then:*

(a) $\mathrm{H}^i\big(X, \mathcal{O}_X(r)\big) = 0$ *for* $0 < i < n$ *and all* $r \in \mathbb{Z}$,

(b) $\mathrm{H}^n\big(X, \mathcal{O}_X(-n-1)\big) \simeq A$,

(c) *the natural map*

$$\mathrm{H}^0\big(X, \mathcal{O}_X(r)\big) \times \mathrm{H}^n\big(X, \mathcal{O}_X(-r-n-1)\big) \to \mathrm{H}^n\big(X, \mathcal{O}_X(-n-1)\big) \simeq A$$

is a perfect pairing of finitely generated free A-modules, for each $r \in \mathbb{Z}$.

The last theorem is important because, most of the time, our framework is to have a product and two projections as in the following diagram:

$$\begin{tikzcd} & X \times \mathbb{P}^n \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & \mathbb{P}^n \end{tikzcd}$$

where X is a quasi-projective variety. In this setting, we consider push forwards $p_{1*}(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})$ for $\mathcal{F}$ a locally free sheaf of $\mathcal{O}_X$-modules and $\mathcal{G}$ a locally free sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules and we compute the right derived functors $R^i p_{1*}(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G})$ with the following result.

Proposition 3.1.6. *Let X be a quasi projective variety and consider the product $X \times \mathbb{P}^n$ as in the previous diagram. Then, given a locally free sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules and $\mathcal{G}$ a locally free sheaf of $\mathcal{O}_{\mathbb{P}^n}$-modules, we have:*

$$\mathrm{R}^i p_{1*}(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}) = \mathcal{F} \otimes \mathrm{H}^i(\mathbb{P}^n, \mathcal{G})$$

Proof. First we use the *projection formula* [Har77, Ex.8.3] stating that

$$\mathrm{R}^i p_{1*}(p_1^*\mathcal{F} \otimes p_2^*\mathcal{G}) = \mathcal{F} \otimes \mathrm{R}^i p_{1*} p_2^*\mathcal{G}$$

We show now that $\mathrm{R}^i p_{1*} p_2^*\mathcal{G} = \mathrm{H}^i(\mathbb{P}^n, \mathcal{G}) \otimes \mathcal{O}_X$. So we consider the following cartesian square

$$\begin{tikzcd} X \times \mathbb{P}^n \arrow[r, "p_2"] \arrow[d, "p_1"'] & \mathbb{P}^n \arrow[d, "f"] \\ X \arrow[r, "g"'] & \mathrm{Spec(k)} \end{tikzcd}$$

By [Har77, III.9.3], there is a natural isomorphism

$$\mathrm{H}^i(\mathbb{P}^n, \mathcal{G}) \otimes \mathcal{O}_X = g^*\mathrm{R}^i f_*\mathcal{G} \simeq \mathrm{R}^i p_{1*} p_2^*\mathcal{G}$$

since X is flat over k. Now since $\mathrm{R}^i f_*\mathcal{G} = \mathrm{H}^i(\mathbb{P}^n, \mathcal{G})$, we have

$$g^*\mathrm{R}^i f_*\mathcal{G} \simeq \mathrm{H}^i(\mathbb{P}^n, \mathcal{G}) \otimes \mathcal{O}_X.$$

□

3.1.2 The Eagon-Northcott complex

As an illustration of Subsection 3.1.1, we construct the resolution of the ideal of the twisted cubic of $\mathbb{P}^3$. Recall that the twisted cubic $\mathcal{C} = \mathbb{V}\big(\mathcal{I}_2(M)\big)$ is the zero locus of the 2×2-minors of the matrix $M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$ with entries in the polynomial ring $R = \mathrm{k}[x_0, \ldots, x_n]$.

As we explained in Subsection 2.1.1, we identify $\mathcal{C}$ with a complete intersection scheme $\mathbb{D}$ in the product $\mathbb{P} = \mathbb{P}^3 \times \mathbb{P}^1$, $\mathbb{P}^3 \times \mathbb{P}^1$ being the projectivization $\mathbb{P}\big(\mathcal{O}^2_{\mathbb{P}^3}\big)$. Hence a locally free resolution of $\mathcal{O}_{\mathbb{D}}$ reads:

$$0 \to \mathcal{O}_{\mathbb{P}}(-3,-3) \to \mathcal{O}_{\mathbb{P}}(-2,-2)^3 \to \mathcal{O}_{\mathbb{P}}(-1,-1)^3 \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{D}} \to 0. \tag{3.1.2}$$

The sheaves $\mathcal{O}_{\mathbb{P}}(-i,-j)$ are the sheaves $p_1^*\mathcal{O}_{\mathbb{P}^3}(-i) \otimes p_2^*\mathcal{O}_{\mathbb{P}^1}(-j)$ of Subsection 3.1.1 where p_1 and p_2 are the projections:

$$\begin{tikzcd} & \mathbb{P}^3 \times \mathbb{P}^1 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbb{P}^3 & & \mathbb{P}^1 \end{tikzcd}$$

Let us now decompose the push forward by p_1 of the exact sequence (3.1.2). In order to do so, we denote as follows the kernel and cokernel in (3.1.2)

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{O}_{\mathbb{P}}(-3,-3) \arrow[r] & \mathcal{O}_{\mathbb{P}}(-2,-2)^3 \arrow[r] \arrow[dr] & \mathcal{O}_{\mathbb{P}}(-1,-1)^3 \arrow[r] \arrow[dr] & \mathcal{O}_{\mathbb{P}} \arrow[r] & \mathcal{O}_{\mathbb{D}} \arrow[r] & 0 \\ & & & \mathcal{G}_2 \arrow[ur] \arrow[dr] & & \mathcal{G}_1 \arrow[ur] \arrow[dr] & \\ & & 0 \arrow[ur] & & 0 \arrow[ur] & & 0 \end{tikzcd}$$

so that $\mathcal{G}_1$ and $\mathcal{G}_2$ are the kernel of (respectively) the first and second homomorphism of (3.1.2).

Now, we apply p_{1*} to the exact sequence:

$$0 \to \mathcal{G}_1 \to \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{D}} \to 0$$

to obtain

$$0 \to p_{1*}\mathcal{G}_1 \to p_{1*}\mathcal{O}_{\mathbb{P}} \to p_{1*}\mathcal{O}_{\mathbb{D}} \to \mathrm{R}^1 p_{1*}\mathcal{G}_1.$$

As we explained in Subsection 2.1.1, $p_{1*}\mathcal{O}_{\mathbb{D}} \simeq \mathcal{O}_{\mathcal{C}}$ and $p_{1*}\mathcal{O}_{\mathbb{P}} \simeq \mathcal{O}_{\mathbb{P}^3}$ so if $\mathrm{R}^1 p_{1*}\mathcal{G}_1 = 0$, a locally free resolution of $\mathcal{O}_{\mathcal{C}}$ is given by a locally free resolution of $p_{1*}\mathcal{G}_1$. We continue this computation by applying p_{1*} to the exact sequence:

$$0 \to \mathcal{G}_2 \to \mathcal{O}_{\mathbb{P}}(-1,-1)^3 \to \mathcal{G}_1 \to 0.$$

We obtain

$$0 \longrightarrow p_{1*}\mathcal{G}_2 \longrightarrow p_{1*}\mathcal{O}_{\mathbb{P}}(-1,-1)^3 \longrightarrow p_{1*}\mathcal{G}_1 \longrightarrow \mathrm{R}^1 p_{1*}\mathcal{G}_2$$

$$\longrightarrow \mathrm{R}^1 p_{1*}\mathcal{O}_{\mathbb{P}}(-1,-1)^3 \longrightarrow \mathrm{R}^1 p_{1*}\mathcal{G}_1 \longrightarrow \mathrm{R}^2 p_{1*}\mathcal{G}_2.$$

Now we focus on $p_{1*}\mathcal{O}_{\mathbb{P}}(-1,-1)^3$ and $\mathrm{R}^1 p_{1*}\mathcal{O}_{\mathbb{P}}(-1,-1)^3$. By Proposition 3.1.6, $p_{1*}\mathcal{O}_{\mathbb{P}}(-1,-1)^3 \simeq \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathrm{H}^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)\big)$ and $\mathrm{H}^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)\big) = 0$ so

$$p_{1*}\mathcal{O}_{\mathbb{P}}(-1,-1)^3 = 0.$$

In the same way,

$$\mathrm{R}^1 p_{1*}\mathcal{O}_{\mathbb{P}}(-1,-1)^3 \simeq \mathcal{O}_{\mathbb{P}^3}(-1) \otimes \mathrm{H}^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)\big)$$

and by Theorem 3.1.5 (c),

$$\mathrm{H}^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-1)\big) = \mathrm{H}^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-(1-2)-2)\big) \simeq \mathrm{H}^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1-2)\big) = 0.$$

Hence, we have that $p_{1*}\mathcal{G}_2 = 0$ and $p_{1*}\mathcal{G}_1 \simeq \mathrm{R}^1 p_{1*}\mathcal{G}_2$. Moreover, if $\mathrm{R}^2 p_{1*}\mathcal{G}_2 = 0$, we have that $\mathrm{R}^1 p_{1*}\mathcal{G}_1 = 0$.

Now we apply p_{1*} to the exact sequence

$$0 \to \mathcal{O}_{\mathbb{P}}(-3,-3) \to \mathcal{O}_{\mathbb{P}}(-2,-2)^3 \to \mathcal{G}_2 \to 0$$

and we obtain

$$0 \longrightarrow p_{1*}\mathcal{O}_{\mathbb{P}}(-3,-3) \longrightarrow p_{1*}\mathcal{O}_{\mathbb{P}}(-2,-2)^3 \longrightarrow p_{1*}\mathcal{G}_2$$

$$\to \mathrm{R}^1 p_{1*}\mathcal{O}_{\mathbb{P}}(-3,-3) \to \mathrm{R}^1 p_{1*}\mathcal{O}_{\mathbb{P}}(-2,-2)^3 \to \mathrm{R}^1 p_{1*}\mathcal{G}_2$$

$$\to \mathrm{R}^2 p_{1*}\mathcal{O}_{\mathbb{P}}(-3,-3) \to \mathrm{R}^2 p_{1*}\mathcal{O}_{\mathbb{P}}(-2,-2)^3 \to \mathrm{R}^2 p_{1*}\mathcal{G}_2$$

$$\to \mathrm{R}^3 p_{1*}\mathcal{O}_{\mathbb{P}}(-3,-3).$$

In this sequence, $p_{1*}\mathcal{G}_2 = 0$ as we have already explained. We compute the sheaf $\mathrm{R}^1 p_{1*}\mathcal{O}_{\mathbb{P}}(-3,-3)$ as follows. By Proposition 3.1.6,

$$\mathrm{R}^1 p_{1*}\mathcal{O}_{\mathbb{P}}(-3,-3) \simeq \mathcal{O}_{\mathbb{P}^3}(-3) \otimes \mathrm{H}^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3)\big)$$

and by Theorem 3.1.5 (c), $\mathrm{H}^1\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-3)\big) \simeq \mathrm{H}^0\big(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(1)\big)$ which has dimension 2. So

$$\mathrm{R}^1 p_{1*}\mathcal{O}_{\mathbb{P}}(-3,-3) \simeq \mathcal{O}_{\mathbb{P}^3}(-3)^2$$

and, in the same way,

$$\mathrm{R}^1 p_{1*}\mathcal{O}_{\mathbb{P}}(-2,-2)^3 \simeq \mathcal{O}_{\mathbb{P}^3}(-2)^3.$$

Using Theorem 3.1.4, we compute also that the higher direct image sheaves vanish. Hence, a locally free resolution of $\mathrm{R}^1 p_{1*}\mathcal{G}_2$ reads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^2 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^3 \longrightarrow \mathrm{R}^1 p_* \mathcal{G}_2 \longrightarrow 0$$

so that a locally free resolution of $\mathcal{O}_{\mathcal{C}}$ is

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^2 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-2)^3 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \mathcal{O}_{\mathcal{C}} \longrightarrow 0.$$

Actually, this situation consisting of pushing forward a Koszul complex in order to describe a resolution of the coordinate ring of a determinantal variety is much more general. The final complex is called the *Eagon-Northcott* complex. We give its definition following [Eis05, A2H].

Let R be a ring and let $F = R^f$ and $G = R^g$ be two free R-modules. The *Eagon-Northcott complex* of a map $\alpha : F \to G$ (or a matrix A representing α) is a complex

$$\begin{aligned} \mathrm{EN}(\alpha): \quad 0 \longrightarrow \mathrm{Sym}_{f-g} G^\vee \otimes \wedge^f F \xrightarrow{d_{f-g+1}} \mathrm{Sym}_{f-g-1} G^\vee \otimes \wedge^{f-1} F \xrightarrow{d_{f-g}} \\ \cdots \longrightarrow \mathrm{Sym}_2 G^\vee \otimes \wedge^{g+2} F \xrightarrow{d_3} G^\vee \otimes \wedge^{g+1} F \xrightarrow{d_2} \wedge^g F \xrightarrow{\wedge^g \alpha} \wedge^g G. \end{aligned}$$

Here $\mathrm{Sym}_k G$ is the k-th symmetric power (or divided power in the case that k has positive characteristic, see[Eis95, A.2.4]) of G.

The homomorphisms d_j are defined as follows. First, we define a diagonal map

$$\Delta : \mathrm{Sym}_k G^\vee \to G^\vee \otimes \mathrm{Sym}_{k-1} G^\vee$$

as the dual of the multiplication map $G \otimes \mathrm{Sym}_{k-1} G \to \mathrm{Sym}_k G$ in the symmetric algebra of G. Next we define an analogous map

$$\Delta : \wedge^k F \to F \otimes \wedge^{k-1} F$$

as the dual of the multiplication in the exterior algebra of $F^\vee$. On decomposable elements, this diagonal has the simple form

$$f_1 \wedge \ldots \wedge f_k \mapsto \sum_i (-1)^{i-1} f_i \otimes f_1 \wedge \ldots \wedge \hat{f}_i \wedge \ldots \wedge f_k.$$

For $u \in \mathrm{Sym}_{j-1} G^\vee$ we write $\Delta(u) = \sum_i u_i' \otimes u_i'' \in G^\vee \otimes \mathrm{Sym}_{j-2} G^\vee$ and similarly for $v \in \wedge^{g+j-1} F$ we write $\Delta(v) = \sum_t v_t' \otimes v_t'' \in F \otimes \wedge^{g+j-2} F$. Note that $\alpha^\vee(u_j') \in F^\vee$ so $[\alpha^\vee(u_i')](v_t') \in R$. We set

$$\begin{aligned} d_j\colon \ \mathrm{Sym}_{j-1} G^\vee \otimes \wedge^{g+j-1} F &\longrightarrow \mathrm{Sym}_{j-2} G^\vee \otimes \wedge^{g+j-2} F \\ u \otimes v &\longmapsto \sum_{s,t} [\alpha^\vee(u_s')](v_t') \cdot u_s'' \otimes v_t'' \end{aligned}$$

We refer to [Eis05, A2H] for the fact that $\mathrm{EN}(\alpha)$ is a complex. The following theorem provides a condition under which $\mathrm{EN}(\alpha)$ is resolution of $\mathrm{coker}(\wedge^g\alpha)$. Recall that the *depth* of an ideal I over a ring R is the length of a maximal regular sequence o R in I.

Theorem 3.1.7. *[Eis05, A.2.60] Let $\alpha : F \to G$ with $\mathrm{rank}(F) = f \geq \mathrm{rank}(G) = g$ be a map of free R-modules of finite rank. The Eagon-Northcott complex $\mathrm{EN}(\alpha)$ is exact (and thus furnishes a free resolution of $R/I_g(\alpha)$) if and only if $\mathrm{depth}\big(I_g(\alpha)\big)$ is equal to $f - g + 1$, the greatest possible value.*

More details about the construction of the Eagon-Northcott complex as a push-forward of a Koszul complex can be found in [Eis95, A2] or [Wey03, 6]. The idea is exactly the one we applied in the case of the twisted cubic of $\mathbb{P}^3$ in the beginning of the subsection. Namely, denoting by $\mathcal{C}$ the scheme $\mathbb{V}\big(\mathcal{I}_g(\alpha)\big)$, we have that $\mathcal{C}$ is isomorphic to a complete intersection $\mathbb{D}$ in $\mathbb{P}(G)$. The push forward of the resolution of $\mathcal{O}_{\mathbb{D}}$, that is a Koszul complex, provides a resolution of $\mathcal{O}_{\mathcal{C}}$. This is the Eagon-Northcott complex resolving $\mathcal{O}_{\mathcal{C}}$.

3.1.3 Gorenstein rings and linkage

To finish this background section, we introduce the notions of *Gorenstein rings* and *linkage*. To this end, we focus one more time on the twisted curve $\mathcal{C}$ in $\mathbb{P}^3$. Recall that the ideal sheaf $\mathcal{I}_{\mathcal{C}}$ of $\mathcal{C}$ is generated by the sections $x_0x_2 - x_1^2, x_0x_3 - x_1x_2, x_1x_3 - x_2^2$ which are the 2×2-minors of the matrix $M = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$.

Now consider the sub-ideal sheaf $\mathcal{I}_{\mathcal{C}\cup\mathrm{L}}$ generated by $x_0x_2 - x_1^2, x_0x_3 - x_1x_2$. It is a computation to show that $\mathbb{V}(\mathcal{I}_{\mathcal{C}\cup\mathrm{L}})$ is the union of the twisted cubic $\mathcal{C}$ and a line $\mathrm{L} = \mathbb{V}(x_0, x_1)$ in $\mathbb{P}^3$. Actually, $\mathcal{C}$ and L meet in one point p of multiplicity 2 (i.e. $\mathrm{length}(\mathcal{O}_{\mathcal{C}\cup\mathrm{L},p}) = 2$). In this case, since the union $\mathcal{C} \cup \mathrm{L}$ is a complete intersection curve, we say that $\mathcal{C}$ and L are *linked*. What is remarkable in this example is that the quotient ideal $(\mathcal{I}_{\mathcal{C}\cup\mathrm{L}} : \mathcal{I}_{\mathcal{C}})$ is equal to $\mathcal{I}_{\mathrm{L}}$ and $(\mathcal{I}_{\mathcal{C}\cup\mathrm{L}} : \mathcal{I}_{\mathrm{L}}) = \mathcal{I}_{\mathcal{C}}$. This property of linkage of $\mathcal{I}_{\mathcal{C}\cup\mathrm{L}}$ is a property shared by any complete intersection scheme. It is the property of being Gorenstein.

Let us now give more precise definitions and results concerning this domain following [Eis95, 21].

Definition 3.1.8. Let A be a local noetherian ring of Krull dimension n, $\mathfrak{m}$ its maximal ideal and k its residue field. A is called *Gorenstein* if

$$\mathrm{Ext}^i_A(\mathrm{k}, A) = 0 \text{ for all } i \neq n \text{ and } \mathrm{Ext}^n(\mathrm{k}, A) \simeq \mathrm{k}.$$

A scheme is called *Gorenstein* if all its local rings are Gorenstein.

Definition 3.1.9. Let A be a local noetherian ring of Krull dimension n, $\mathfrak{m}$ its maximal ideal and k its residue field. An A-module M such that the module $\mathrm{Ext}^n_A(A/\mathfrak{m}, M)$ vanishes if $n \neq \mathrm{height}(\mathfrak{m})$ and is 1-dimensional if $n = \mathrm{height}(\mathfrak{m})$ is called the *canonical module of* A. In this case, we denote ω_A the canonical module M.

As a first consequence, we have.

Proposition 3.1.10. *In the situation where A has a canonical module, A is Gorenstein if and only if $\omega_A = A$. Given a Gorenstein scheme X, the invertible sheaf associated to the local canonical modules is called the* dualizing sheaf of X *and denoted by ω_X.*

Theorem 3.1.11. *[Eis95, 21.19] Any complete intersection scheme is Gorenstein.*

As we will see, determinantal schemes are also Gorenstein under certain conditions. For us, the result we will use the most concerning Gorenstein schemes is Theorem 3.1.14 below. Recall that the codimension of an ideal I over a ring A is the Krull dimension of the localization A_I [Eis95, 9]. We define also Cohen-Macaulay ideal.

Definition 3.1.12. A ring R is *Cohen-Macaulay* if $\operatorname{depth}(I) = \operatorname{codim}(I)$ for every ideal I of R.

Proposition 3.1.13. *[BH93] Gorenstein rings are Cohen-Macaulay.*

Theorem 3.1.14. *[Eis95, 21.23] Let A be a Gorenstein local ring, and let I be an ideal of codimension 0. Set $B = A/I$ and $J = (0 :_A I)$. We have $J \simeq \operatorname{Hom}_A(B, A)$.*

(a) *The ideal J has codimension 0 and no embedded components. If I has no embedded components, then $I = (0 :_A J)$ so I and J are linked.*

(b) *If $B = A/I$ is a Cohen-Macaulay ring, then $C = A/J$ is a Cohen-Macaulay ring.*

(c) *If $B = A/I$ is a Cohen-Macaulay ring, then $J = (0 :_A I) = \operatorname{Hom}_A(B, A)$ is a canonical module for B; in particular, B is Gorenstein if and only if J is a principal ideal of A.*

To finish this section about Gorenstein properties, let us explain in more details some facts we will use in the following.

Proposition 3.1.15. *[Eis95, Theorem 21.15] Let $\mathbb{P}$ be a smooth variety, $\omega_{\mathbb{P}}$ be its dualizing sheaf and let $\mathbb{K}$ be a Gorenstein subscheme of $\mathbb{P}$ of codimension n. Then the dualizing sheaf $\omega_{\mathbb{K}}$ of $\mathbb{K}$ verifies $\omega_{\mathbb{K}} \simeq \mathcal{E}\mathrm{xt}^n(\mathcal{O}_{\mathbb{K}}, \omega_{\mathbb{P}})$.*

Remark 3.1.16. In the settings of Proposition 3.1.15, let

$$\begin{tikzcd}
0 \arrow[r] & \mathfrak{Q}_n \arrow[r] & \mathfrak{Q}_{n-1} \arrow[r] \arrow[dr] & \mathfrak{Q}_{n-2} \arrow[r] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & \mathfrak{Q}_0 \arrow[r] & \mathcal{O}_{\mathbb{K}} \arrow[r] & 0 \\
& & & \mathcal{E}_{n-1} \arrow[ur] & \mathcal{E}_{n-1} \arrow[ur] & \mathcal{E}_1 \arrow[ur] & &
\end{tikzcd}$$

be a locally free resolution of $\mathcal{O}_{\mathbb{K}}$, it has length n, since, as a Gorenstein scheme, $\mathbb{K}$ is Cohen-Macaulay. By this, we set that the sheaves $\mathfrak{Q}_i$ are the locally free sheaves of the resolution and the sheaves $\mathcal{E}_i$ are the kernels and cokernels of the corresponding morphisms. Then, a locally free resolution of $\omega_{\mathbb{K}}$ reads:

$$0 \longrightarrow \mathfrak{Q}_0^\vee \longrightarrow \mathfrak{Q}_1^\vee \longrightarrow \mathfrak{Q}_2^\vee \longrightarrow \cdots \longrightarrow \mathfrak{Q}_n^\vee \longrightarrow \omega_{\mathbb{K}} \longrightarrow 0.$$

Indeed, applying the functor $\mathcal{H}\mathrm{om}(\cdot, \omega_{\mathbb{P}})$ successively on each exact sequence

$$0 \longrightarrow \mathcal{E}_i \longrightarrow \mathfrak{Q}_{i-1} \longrightarrow \mathcal{E}_{i-1} \longrightarrow 0$$

we can compute by cohomology chasing that $\mathcal{E}\mathrm{xt}^1(\mathcal{E}_{n-1}, \omega_{\mathbb{P}}) \simeq \mathcal{E}\mathrm{xt}^n(\mathcal{O}_{\mathbb{K}}, \omega_{\mathbb{P}}) \simeq \omega_{\mathbb{K}}$. This is because the derived sheaves $\mathcal{E}\mathrm{xt}^j(\mathfrak{Q}_i, \omega_{\mathbb{P}})$ vanishes for all $i \in \{0, \ldots, n\}$ and all $j > 0$ since the sheaves $\mathfrak{Q}_i$ are locally free.

3.2 Subregularity of the symmetric algebra

3.2.1 Generalities

Assuming that the subscheme $Z = \mathbb{V}(\mathcal{I}_Z)$ of X defined by $\mathcal{I}_Z$ is zero-dimensional in a quasi-projective variety X, we provide here a resolution of the ideal of $\mathbb{X} = \mathbb{P}(\mathcal{I}_Z)$ in terms of the pulled-back resolution of the dualizing module of Z, up to some shift in degree, and of the Eagon-Northcott complex associated with another still larger algebra, which we call the Koszul hull.

Our precise settings are as follows unless otherwise specified.

Notation 3.2.1. Put $n = \dim(X)$ and let $\mathcal{I}_Z$ be an ideal sheaf generated by $n+1$ global sections $\phi_0, \ldots, \phi_n$ of a vector bundle $\mathcal{L}$ over X. We assume that the subscheme $Z = \mathbb{V}(\mathcal{I}_Z)$ of X is 0-dimensional. Letting V be the $(n+1)$-dimensional subspace generated by $\phi_0, \ldots, \phi_n$, we consider a surjection

$$\mathrm{V} \otimes \mathcal{L}^\vee \to \mathcal{I}_Z \to 0 \tag{3.2.1}$$

which provides a closed embedding $\mathbb{X} \hookrightarrow \mathbb{P} \simeq \mathbb{P}^n_X$ of $\mathbb{X} = \mathbb{P}(\mathcal{I}_Z)$.

Here, we denote by $\mathbb{P}$ the projective bundle

$$\mathbf{Proj}\left(\mathrm{Sym}(\mathcal{O}_X(-\eta)^{n+1})\right)$$

where η stands for $c_1(\mathcal{L})$. We also consider its bundle map $p : \mathbb{P} \to X$ and we let $y_0, \ldots, y_n$ be its relative homogeneous coordinates.

Depending on the context η can stand also for $p^*c_1(\mathcal{L})$ and we let ξ be the first Chern class of $\mathcal{O}_{\mathbb{P}}(1)$.

Now, put

$$\mathfrak{Q}_{i,j} = (\overset{i+1}{\wedge} \mathrm{V}) \otimes \mathcal{O}_{\mathbb{P}}(-(j+1)\xi - (i-j)\eta) \quad \text{for } 1 \leq i \leq n \text{ and } 0 \leq j \leq i-1$$

and $\mathfrak{Q}_i = \bigoplus_{j=0}^{i-1} \mathfrak{Q}_{i,j}$. The sheaves $\mathfrak{Q}_i$ are the terms of the Eagon-Northcott complex associated with the map

$$\psi : \mathrm{V} \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(\eta) \oplus \mathcal{O}_{\mathbb{P}^n_X}(\xi).$$

defined by the matrix

$$\begin{pmatrix} \phi_0 & \ldots & \phi_n \\ y_0 & \ldots & y_n \end{pmatrix}.$$

The complex takes the form:

$$0 \longrightarrow \mathfrak{Q}_n \longrightarrow \dots \longrightarrow \mathfrak{Q}_1 \longrightarrow \mathcal{O}_{\mathbb{P}^n_X}. \qquad (\mathfrak{Q}_\bullet)$$

Since $\dim(Z) = 0$, let:

$$0 \longrightarrow \mathcal{P}_n \longrightarrow \dots \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{O}_Z \longrightarrow 0 \qquad (\mathcal{P}_\bullet)$$

be a locally free resolution of $\mathcal{O}_Z$, so here $\mathcal{P}_0 = \mathcal{O}_X$ and $\mathcal{P}_1 = \mathrm{V} \otimes \mathcal{L}^\vee$. We emphasize that the length of this resolution is $n = \dim(X)$ since the scheme Z is locally Cohen-Macaulay.

Set

$$\mathcal{P}'_i = p^*\mathcal{P}^\vee_{n+1-i} \otimes \mathcal{O}_{\mathbb{P}}(-n\eta - \xi) \quad \text{for } 1 \leq i \leq n+1$$

and let $\mathcal{I}_{\mathbb{X}}$ be the ideal of $\mathbb{X}$ into $\mathbb{P}$. Our result is the following:

Theorem 3.2.2. *Under the assumption that* $\dim(Z) = 0$, $\mathbb{X}$ *is Cohen-Macaulay of dimension* n *and there is a locally free resolution of* $\mathcal{I}_{\mathbb{X}}$ *of the following form:*

$$0 \longrightarrow \mathcal{P}'_{n+1} \longrightarrow \begin{matrix} \mathfrak{Q}_n \\ \oplus \\ \mathcal{P}'_n \end{matrix} \longrightarrow \dots \longrightarrow \begin{matrix} \mathfrak{Q}_1 \\ \oplus \\ \mathcal{P}'_1 \end{matrix} \longrightarrow \mathcal{I}_{\mathbb{X}} \longrightarrow 0. \qquad \text{(R1)}$$

Denoting by y_i the homogeneous relative coordinates of the projective bundle $\mathbb{P}$, we make the following definition.

Definition 3.2.3. A complex $(\mathcal{R}_\bullet)$ over $\mathbb{P}$ is *subregular* if for all i the differential $\mathcal{R}_i \to \mathcal{R}_{i-1}$ is linear or constant in the y variables.

In the proof of Theorem 3.2.2 page 70, we will see that the differentials of R1 are actually linear or constant in the y variables so, anticipating, we can state that:

Corollary 3.2.4. *The ideal* $\mathcal{I}_{\mathbb{X}}$ *admits a subregular locally free resolution over* $\mathbb{P}$.

In the last subsection, we focus on a graded version of this result. The motivation to study specifically this case is that it is the framework when working with a rational map from $\mathbb{P}^n$ to $\mathbb{P}^n$. It is moreover the case in which we have a more refined minimal free resolution of the symmetric algebra. For instance

Example 3.2.5. Let $I_Z = (x_1^2 - x_1x_3, x_2^2 - x_2x_3, x_1x_2, x_0x_3)$ be the ideal over $R = \mathrm{k}[x_0, \dots, x_3]$ (remark that that $Z = \mathbb{V}(I_Z)$ is indeed 0-dimensional in $\mathbb{P}^3$).

By a computation via Macaulay2, we have that a minimal free resolution of I_Z reads

$$0 \longrightarrow R(-5)^2 \longrightarrow \begin{matrix} R(-3)^2 \\ \oplus \\ R(-4)^3 \end{matrix} \longrightarrow R(-2)^4 \longrightarrow I_Z \longrightarrow 0.$$

Now, take the last two modules of this complex and tensor them by $S(2, -1)$ where $S = R[y_0, \dots, y_3]$. They give the beginning of a minimal free resolution of the ideal $I_{\mathbb{X}}$ of the symmetric algebra of I_Z over S (also computed with Macaulay2)

$$0 \longrightarrow \begin{matrix} S(-4,-2) \\ \oplus \\ S(-2,-3) \end{matrix} \longrightarrow \begin{matrix} S(-2,-2)^4 \\ \oplus \\ S(-3,-1)^2 \end{matrix} \longrightarrow \begin{matrix} S(-1,-1)^2 \\ \oplus \\ S(-2,-1)^3 \end{matrix} \longrightarrow I_{\mathbb{X}} \longrightarrow 0.$$

More generally take $R = \mathrm{k}[x_0, \dots, x_n]$ and $I_Z = (\phi_0, \dots, \phi_n)$ an ideal over R generated by $n+1$ homogeneous polynomials of degree η (this integer η should be understood as the degree of $c_1(\mathcal{L})$ in the sheafified case assuming $X = \mathbb{P}^n$). The ideal of the symmetric algebra of I_Z, denoted by $I_{\mathbb{X}}$, is a bigraded homogeneous ideal of $S = R[y_0, \dots, y_n]$. This time we consider the two complexes $(P'_\bullet)$ and $(Q_\bullet)$ obtained by taking the graded modules of global sections of $(\mathcal{P}'_\bullet)$ and $(\mathfrak{Q}_\bullet)$. These are S-graded subregular complexes. Our result in this setting is the following.

Theorem 3.2.6. *Assume I_Z is a graded homogeneous Cohen-Macaulay ideal of dimension 1, then $\mathbb{X}$ is Cohen-Macaulay and a minimal bigraded S-free resolution of $I_{\mathbb{X}}$ reads:*

$$0 \longrightarrow Q''_n \longrightarrow \begin{matrix} Q''_{n-1} \\ \oplus \\ P''_{n-1} \end{matrix} \longrightarrow \begin{matrix} Q''_{n-2} \\ \oplus \\ P''_{n-2} \end{matrix} \longrightarrow \dots \longrightarrow \begin{matrix} Q''_2 \\ \oplus \\ P''_2 \end{matrix} \longrightarrow P''_1 \longrightarrow I_{\mathbb{X}} \longrightarrow 0 \qquad \text{(R2)}$$

where

$$Q''_i = \bigoplus_{j=1}^{n} Q_{i,j}, \qquad Q_{i,j} = S\big(-(i-j)\eta, -j-1\big)^{\binom{n+1}{i+1}}, \qquad P''_i = P_{i+1} \otimes S(\eta, -1).$$

Moreover Theorem 3.2.2 and Theorem 3.2.6 are sharp in the following sense. If $\dim(Z) > 0$, then the resolution of $\mathbb{X}$ might not be subregular as shown in the following example explained to us by Aldo Conca.

Example 3.2.7. In $\mathbb{P}^3$, consider the zero locus Z of the ideal $I_Z = (-x_2^3 x_3 + x_3^4, -x_2^4 - x_3^4, -x_1 x_3^3 - x_3^4, x_2^2 x_3^2 + x_3^4)$. The ideal I_Z has Krull dimension 2 over $R = \mathrm{k}[x_0, \dots, x_3]$, so $\dim(Z) = 1$, and a minimal graded free resolution of $I_{\mathbb{X}}$ reads:

$$0 \to S(-5,-3) \to \begin{matrix} S(-5,-2) \\ \oplus \\ S(-4,-3)^3 \end{matrix} \to \begin{matrix} S(-4,-1) \\ \oplus \\ S(-3,-2)^3 \\ \oplus \\ S(-4,-2) \\ \oplus \\ S(-3,-3) \end{matrix} \to \begin{matrix} S(-1,-1) \\ \oplus \\ S(-2,-1)^2 \\ \oplus \\ S(-3,-1) \end{matrix} \to I_{\mathbb{X}} \to 0$$

where we wrote the shift in the y variables in the right position. Hence the resolution of $I_{\mathbb{X}}$ is not subregular.

3.2.2 Local resolution of the symmetric algebra

Preliminaries

Recall Proposition 2.1.3 that letting a locally free presentation of $\mathcal{I}_Z$

$$\mathcal{P}_2 \overset{M}{\to} \mathcal{P}_1 \to \mathcal{I}_Z \to 0, \tag{3.2.2}$$

$\mathbb{X}$ is the zero scheme of the corresponding section of the composition map $s \in \mathrm{H}^0\big(\mathbb{P}, p^*\mathcal{P}_2^\vee \otimes \mathcal{O}_{\mathbb{P}}(\xi)\big)$. In other words, the ideal sheaf $\mathcal{I}_{\mathbb{X}}$ of $\mathbb{X}$ in $\mathbb{P}$ is locally generated by the entries of the row matrix $\mathbf{y}p^*M$ where $\mathbf{y}$ stands for $(y_0 \ \dots \ y_n)$. We denote by M_x the matrix obtained from M by specializing at the point $x \in X$.

We emphasize the following remark. Since $\dim(X) = n$ and $\operatorname{codim}(Z, X) = n$, the local ideal sheaf $\mathcal{I}_{Z,z}$ of a point $z \in Z$ is generated by at least n independent sections of $\mathcal{L}$ lying in V. The crucial point is to take care of the case where $z \in Z$ is a point at which Z is not a complete intersection, i.e all the sections $\phi_0, \dots, \phi_n$ are required to generate $\mathcal{I}_{Z,z}$. The following result is restatement of Proposition 2.2.13 but we re-explain its proof in the current settings.

Lemma 3.2.8. *Let $x \in X$ be a closed point. The scheme-theoretic fibre $\mathbb{X}_x$ is:*

(i) a point if $x \notin Z$,

(ii) isomorphic to $\mathbb{P}^{n-1}_x$ if $x \in Z$ and Z is a local complete intersection at x,

(iii) isomorphic to $\mathbb{P}^n_x$ if $x \in Z$ and Z is not a local complete intersection at x.

Proof. Since the formation of the symmetric algebra commutes with base change (see Proposition 2.1.1), the fibre $\mathbb{X}_x$ is obtained by localizing X at x and taking $\mathbb{P}(\mathcal{I}_Z \otimes \mathrm{k}_x)$, where k_x is the residue field of $\mathcal{O}_X$ at x.

(i) If $x \notin Z$, locally at x the ideal $\mathcal{I}_Z$ is just $\mathcal{O}_X$, so p is an isomorphism of $\mathbb{X}_x$ to x.

(ii),(iii) If $x \in Z$, since Z has codimension n in X, a subspace of n independent local sections of $\mathcal{L}$ from V is needed at least to generate $\mathcal{I}_Z$ locally around x. Actually such subspace exists if and only if Z is a local complete intersection (LCI) at x. In other words, $\mathcal{I}_Z \otimes \mathrm{k}_x$ is a k_x-vector space which can be generated by an n-dimensional subspace of V if and only if Z is LCI at x, so that $\mathcal{I}_Z \otimes \mathrm{k}_x$ is isomorphic to k_x^n or to k_x^{n+1} depending on whether Z is LCI at x or not. Therefore $\mathbb{P}(\mathcal{I}_Z \otimes \mathrm{k}_x)$ is isomorphic to $\mathbb{P}^{n-1}_x$ or $\mathbb{P}^n_x$ depending on whether Z is LCI at x or not.

□

Remark 3.2.9. Recall that in our setting of a zero-dimensional scheme Z, the set of points $z \in Z$ such that $\mathbb{X}_z \simeq \mathbb{P}^n_z$ is equal set theoretically to $\mathbb{V}\big(\operatorname{Fitt}_{n-1}(\mathcal{I}_Z)\big)$ where $\operatorname{Fitt}_{n-1}(\mathcal{I}_Z)$ is the ideal generated by the entries of M (Proposition 2.2.19).

Now, we take the Koszul complex with respect to the map $\mathrm{V} \otimes \mathcal{O}_X(-\eta) = \mathcal{P}_1 \xrightarrow{\Phi} \mathcal{O}_X$ and we write $k_i(\Phi)$ for the i-th differential of the Koszul complex. We have the following diagram:

$$\begin{tikzcd}
\wedge^2\mathcal{P}_1 \arrow[rr, "k_1(\Phi)"] \arrow[dr] & & \mathcal{P}_1 \arrow[r, "\Phi"] & \mathcal{I}_Z \arrow[r] & 0 \\
& \mathcal{F}_1 \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{K3}$$

in which the first row is not exact since Z is not empty and where we put $\mathcal{F}_1 = \mathrm{Im}\big(k_1(\Phi)\big)$. By definition of the presentation and the Koszul complex, we have $\mathcal{F}_1 \subset \mathcal{E}$ and $\mathcal{E}/\mathcal{F}_1 = \mathcal{H}_1(\mathcal{I}_Z)$ where $\mathcal{E}$ is the kernel of the morphism $\mathcal{P}_1 \to \mathcal{I}_Z$ and $\mathcal{H}_1(\mathcal{I}_Z)$ stands for the first Koszul homology sheaf of the set $(\phi_0 \ \dots \ \phi_n)$ of generators of $\mathcal{I}_Z$ (see Definition 2.2.5 for the definition of the Koszul homology).

The Koszul hull

We introduce now another subscheme of $\mathbb{P}$ which we call the Koszul hull of $\mathbb{X}$. This subscheme contains $\mathbb{X}$ and actually differs from $\mathbb{X}$ by a copy of $\mathbb{P}^n_Z$, as we will see.

Definition 3.2.10. Set notation as in (K3) and let $\mathcal{I}_{\mathbb{K}}$ be the ideal sheaf generated by the entries in the row matrix $\mathbf{y}p^*k_1(\Phi)$. We call the *Koszul hull*, denoted by $\mathbb{K}$, the subscheme in $\mathbb{P}$ defined by $\mathbb{K} = \mathbb{V}(\mathcal{I}_{\mathbb{K}})$.

Now, we explain the strategy of the proof of Theorem 3.2.2. Via the inclusion $\mathcal{F}_1 \subset \mathcal{E}$, we see that $\mathcal{I}_{\mathbb{K}} \subset \mathcal{I}_{\mathbb{X}}$, that is $\mathbb{X} \subset \mathbb{K}$. Hence we have the following short exact sequence:

$$0 \longrightarrow \mathcal{I}_{\mathbb{K}} \longrightarrow \mathcal{I}_{\mathbb{X}} \longrightarrow \mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}} \longrightarrow 0.$$

So in order to get the subregularity of the resolution of $\mathcal{I}_{\mathbb{X}}$, we first show the subregularity of resolutions of $\mathcal{I}_{\mathbb{K}}$ and of $\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}}$ and from there, we show how we get the resolution of $\mathcal{I}_{\mathbb{X}}$ by patching together these resolutions.

We start by analysing the Koszul hull more closely.

Proposition 3.2.11. *We have the following properties.*

(i) The scheme $\mathbb{K}$ is determinantal. More precisely, $\mathcal{I}_{\mathbb{K}}$ is the ideal of the 2×2 minors of the map $\mathrm{V} \otimes \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(\eta) \oplus \mathcal{O}_{\mathbb{P}}(\xi)$ defined by the matrix:

$$\psi = \begin{pmatrix} \phi_0 & \dots & \phi_n \\ y_0 & \dots & y_n \end{pmatrix}.$$

Under the assumption that $\dim_X(Z) = 0$:

(ii) $\mathrm{codim}(\mathbb{K}, \mathbb{P}) = n$.

(iii) A locally free resolution of $\mathcal{I}_{\mathbb{K}}$ is the sheafification of the Eagon-Northcott complex. Namely, there is a long exact sequence:

$$0 \longrightarrow \mathfrak{Q}_n \longrightarrow \ldots \longrightarrow \mathfrak{Q}_2 \longrightarrow \mathfrak{Q}_1 \longrightarrow \mathcal{I}_{\mathbb{K}} \longrightarrow 0 \qquad (\mathfrak{Q}_\bullet)$$

where $\mathfrak{Q}_i = \bigoplus_{j=0}^{i-1} \mathfrak{Q}_{i,j}$ *and*

$$\mathfrak{Q}_{i,j} = (\overset{i+1}{\wedge} \mathrm{V}) \otimes \mathcal{O}_{\mathbb{P}}(-(j+1)\xi - (i-j)\eta) \quad \text{for } 1 \leq i \leq n \text{ and } 0 \leq j \leq i-1.$$

(iv) The scheme $\mathbb{K}$ is Gorenstein and its dualizing sheaf $\omega_{\mathbb{K}}$ verifies:

$$\omega_{\mathbb{K}} \simeq p^*\omega_X \otimes \mathcal{O}_{\mathbb{P}}(n\eta - n\xi).$$

Proof. *(i)* The morphism $k_1(\Phi)$ takes the form,

$$k_1(\Phi) = \begin{pmatrix} \phi_1 & \phi_2 & \ldots \\ -\phi_0 & 0 & \ldots \\ 0 & -\phi_0 & \ldots \\ \vdots & 0 & \ldots \\ \vdots & \vdots & \ldots \end{pmatrix}$$

and $\mathcal{I}_{\mathbb{K}}$ is generated by the entries in the row matrix $\mathbf{y}p^*k_1(\Phi)$. Those entries are the same as the 2×2 minors of the matrix ψ.

(ii) We argue set-theoretically by looking at the fibres of the map $\mathbb{K} \to X$ obtained as restriction of p to $\mathbb{K}$. First, note that if $z \notin Z$, then it is clear by the definition of $\mathbb{K}$ that $\mathbb{K}_z$ is a single point. On the other hand, if $z \in Z$ then $\phi_i(z) = 0$ for all $i \in \{0, \ldots, n\}$ so by definition of $\mathbb{K}$ we have $\mathbb{K}_z = \mathbb{P}^n_z$. Therefore the reduced structure of $\mathbb{K}$ is the union of X and of $\cup_{z \in Z} \mathbb{P}^n_z$. This proves that $\mathbb{K}$ has dimension n.

(iii) Since $\mathbb{K}$ is determinantal of the expected codimension, it is Cohen-Macaulay [BV88, Cor. 2.8]. Hence $\mathrm{depth}(\mathcal{I}_{\mathbb{K}}) = \mathrm{codim}(\mathbb{K}, \mathbb{P}) = n$. Therefore the Eagon-Northcott complex provides a global resolution of the ideal $\mathcal{I}_{\mathbb{K}}$ [BV88, Th. 2.16]. The first map $\wedge^2 \mathrm{V} \otimes \mathcal{O}_{\mathbb{P}} \to \wedge^2 \mathcal{O}_{\mathbb{P}}(\eta) \oplus \mathcal{O}_{\mathbb{P}}(\xi)$ of the Eagon-Northcott complex is the matrix $\wedge^2\psi$. Hence the complex $(\mathfrak{Q}_\bullet)$ provides a resolution of $\mathcal{I}_{\mathbb{K}}$.

(iv) Refering to Proposition 3.1.15, we know that if $\mathbb{K}$ is Gorenstein, then its dualizing sheaf $\omega_{\mathbb{K}}$ should be isomorphic to $\mathcal{E}\mathrm{xt}^n(\mathcal{O}_{\mathbb{K}}, \omega_{\mathbb{P}})$. So set $\omega_{\mathbb{K}} = \mathcal{E}\mathrm{xt}^n(\mathcal{O}_{\mathbb{K}}, \omega_{\mathbb{P}})$ and let show that this verifies the properties of a dualizing sheaf of a Gorenstein scheme. By the previous item *(iii)*, a resolution of $\omega_{\mathbb{K}}$ is given by:

$$0 \to \mathfrak{Q}_1^\vee \otimes \omega_{\mathbb{P}} \to \ldots \to \mathfrak{Q}_{n-1}^\vee \otimes \omega_{\mathbb{P}} \xrightarrow{M_1} \mathfrak{Q}_n^\vee \otimes \omega_{\mathbb{P}} \to \omega_{\mathbb{K}} \to 0 \qquad (3.2.3)$$

see Remark 3.1.16 for a better explanation of this fact. Hence, the first condition on the vanishing of the sheaves $\mathcal{E}\mathrm{xt}^i(\mathcal{O}_{\mathbb{K}}, \omega_{\mathbb{K}})$ for $i < n$ is verified

locally by localizing (3.2.3) at any $p \in \mathbb{K}$ and noting that $\mathrm{Ext}^i(\mathcal{O}_{\mathbb{K},p}, \omega_{\mathbb{K}}, p)$ vanishes.

We show now that $\omega_{\mathbb{K}}$ is locally free of rank 1. Locally, we can write explicitly the matrix M_1 which is the transpose of the last matrix in the Eagon-Northcott complex. So M_1 has size $n \times (n-1)(n+1)$ and locally takes the form:

$$M_1 = \begin{pmatrix} \phi_0 & \cdots & \phi_n & 0 & \cdots & & & & & & & & & & 0 \\ y_0 & \cdots & y_n & \phi_0 & \cdots & \phi_n & 0 & \cdots & & & & & & & 0 \\ 0 & \cdots & 0 & y_0 & \cdots & y_n & \phi_0 & \cdots & \phi_n & 0 & \cdots & & & & 0 \\ & & & & & & & & & & & & & & \\ 0 & \cdots & & & & & & & & 0 & y_0 & \cdots & & & y_n \end{pmatrix}$$

Consider an open cover of X by a family of open subsets $\{U_t \mid t \in J\}$ such that $U_t \cap Z = \{z_t\}$. If $z \in U \subset U_t \backslash \{z_t\}$ for all t, then the restriction of $\mathcal{I}_Z$ to U is equal to $\mathcal{O}_U$ so that $\mathbb{K}_U = \mathbb{X}_U = U$ is obviously Gorenstein, because U is smooth.

Or else, if $z = z_t$ for some t, then $\phi_s(z) = 0$ for all $s \in \{0, \ldots, n\}$. In this case, since every point $(y_0 : \ldots : y_n) \in \mathbb{P}^n_z$ has at least one non zero coordinate, the matrix $(M_1)_z$ has corank 1. This shows that for any point of $\mathbb{X}$, the stalk of $\omega_{\mathbb{K}}$ has rank 1 at that point, so $\omega_{\mathbb{K}}$ is locally free of rank one. Hence $\mathbb{K}_{U_j}$ is Gorenstein. This proves that $\mathbb{K}$ is Gorenstein.

Now, we show the isomorphism

$$\omega_{\mathbb{K}} \simeq p^* \omega_X \otimes \mathcal{O}_{\mathbb{P}}(n\eta - n\xi).$$

To do this, we first give an explicit formula for $\omega_{\mathbb{K}}$ by describing the scheme $\mathbb{K}$ as a complete intersection into a larger projective bundle (see [Ein93] for more details about this construction). Let $\mathbb{B}$ be the projective bundle $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}}(\eta) \oplus \mathcal{O}_{\mathbb{P}}(\xi)\big)$ and put ζ for the relative hyperplane class of the bundle map $q : \mathbb{B} \to \mathbb{P}$. A divisor D in $|\mathcal{O}_{\mathbb{B}}(\zeta)|$ corresponds to a morphism $\psi_D : \mathcal{O}_{\mathbb{P}} \to \mathcal{O}_{\mathbb{P}}(\eta) \oplus \mathcal{O}_{\mathbb{P}}(\xi)$. Since the matrix ψ whose 2×2 minors define $\mathbb{K}$ has constant rank 1 over $\mathbb{K}$, the map q restricts to an isomorphism from the complete intersection $\cap_{i=0}^n D_i$ to $\mathbb{K}$, where D_i corresponds to $\psi_{D_i} = (\phi_i, y_i)$.

Therefore, by adjunction we have:

$$q^* \omega_{\mathbb{K}} \simeq \omega_{\mathbb{B}}\big((n+1)\zeta\big). \tag{3.2.4}$$

Next, we show that:

$$\mathcal{O}_{\mathbb{K}}(\zeta) \simeq \mathcal{O}_{\mathbb{K}}(\eta). \tag{3.2.5}$$

Indeed, given a divisor $D \in |\mathcal{O}_{\mathbb{B}}(\zeta)|$, the intersection $D \cap \mathbb{K}$ is defined in $\mathbb{P}$ by the vanishing of the 2×2 minors of the matrix:

$$\begin{pmatrix} \phi_0 & \ldots & \phi_n & \phi_D \\ y_0 & \ldots & y_n & y_D \end{pmatrix},$$

where $\psi_D = (\phi_D, y_D)$ corresponds to D. Since y_D lies in $\langle y_0, \dots, y_n \rangle$, this matrix is equivalent up to row and column operations to:

$$\begin{pmatrix} \phi_0 & \dots & \phi_n & \phi'_D \\ y_0 & \dots & y_n & 0 \end{pmatrix},$$

for some $\phi'_D \in \mathrm{H}^0(X, \mathcal{L})$.

This means that the ideal of $D \cap \mathbb{K}$ in $\mathbb{K}$ is generated by $(y_0\phi'_D, \dots, y_n\phi'_D)$. Since all the y_i do not vanish simultaneously, this implies that $\mathcal{O}_{\mathbb{K}}(\xi)$ is generated by the restriction to $\mathbb{K}$ of ϕ'_D. Hence $\mathcal{O}_{\mathbb{K}}(\zeta) \simeq \mathcal{O}_{\mathbb{K}}(\eta)$ and we compute:

$$\omega_{\mathbb{P}} \simeq p^*\omega_X \otimes \mathcal{O}_{\mathbb{P}}\big(-(n+1)\xi\big)$$

and therefore:

$$\omega_{\mathbb{B}} \simeq q^*\omega_{\mathbb{P}} \otimes \mathcal{O}_{\mathbb{B}}(-2\zeta + \eta + \xi).$$

Hence by (3.2.4) and (3.2.5), we get that $\omega_{\mathbb{K}} \simeq p^*\omega_X \otimes \mathcal{O}_{\mathbb{P}}(n\eta - n\xi)$.

□

From the description of the morphisms d_i of the Eagon-Northcott complex given just before Theorem 3.1.7, we have that these morphisms have constant or linear entries in the y_i variables. Hence

Corollary 3.2.12. *The resolution* $(\mathfrak{Q}_\bullet)$ *is subregular.*

Description of the quotient $\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}}$

We show now the subregularity of a locally free resolution of the quotient $\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}}$. Let us outline that since Z is a zero dimensional scheme of finite type it is affine hence projective. As such it has a dualizing sheaf ω_Z, see [Har77, Proposition 7.5].

Proposition 3.2.13. *We have the following isomorphism:*

$$\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}} \simeq p^*(\omega_Z \otimes \omega_X^\vee) \otimes \mathcal{O}_{\mathbb{P}}(-n\eta - \xi).$$

The proof of this proposition is the object of Lemma 3.2.14. Its proof and the proof of Proposition 3.2.13 rely mostly on Theorem 3.1.14.

Lemma 3.2.14. *The quotient ideal sheaf* $(\mathcal{I}_{\mathbb{K}} : \mathcal{I}_{\mathbb{X}})$ *is isomorphic to* $p^*\mathcal{I}_Z$.

Proof. As in the proof of Proposition 3.2.11, we denote by $k_1(\mathbf{y})$ the first differential in the Koszul complex associated to the map $(y_0 \ \dots \ y_n)$. We denote also by $\mathcal{I}_{\mathbb{X},\mathbb{K}}$ the ideal of $\mathbb{X}$ in $\mathbb{K}$ and $\mathbb{W}$ stands for the scheme p^*Z. Of course we have $\mathbb{W} \simeq \mathbb{P}^n_Z$. The inclusion $\mathcal{I}_{\mathbb{K}} \subset \mathcal{I}_{\mathbb{W}}$ explains the right horizontal exact sequence in the following commutative diagram:

$$\begin{tikzcd}
 & & & 0 \arrow[d] & \\
 & & \wedge^2 \mathrm{V} \otimes \mathcal{O}_{\mathbb{P}}(-\xi-\eta) \arrow[r, "\mathbf{y}p^*k_1(\Phi)"] \arrow[d, "k_1(\mathbf{y})"] & \mathcal{I}_{\mathbb{K}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & p^*\mathcal{E} \arrow[r, "p^*M"] \arrow[d] \arrow[dr, dashed, "\beta"] & \mathrm{V} \otimes \mathcal{O}_{\mathbb{P}}(-\eta) \arrow[r, "p^*\Phi"] \arrow[d, "\mathbf{y}"] & \mathcal{I}_{\mathbb{W}} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{C} \arrow[r] \arrow[d] & \mathcal{O}_{\mathbb{P}}(\xi-\eta) \arrow[r] \arrow[d] & \mathcal{I}_{\mathbb{W}}/\mathcal{I}_{\mathbb{K}} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

The commutativity in the right above square comes from the following fact. Writing down the matrix $k_1(\mathbf{y})$ as follows:

$$k_1(\mathbf{y}) = \begin{pmatrix} y_1 & y_2 & \dots \\ -y_0 & 0 & \dots \\ 0 & -y_0 & \dots \\ \vdots & 0 & \dots \\ \vdots & \vdots & \dots \end{pmatrix}$$

and similarly for $k_1(\Phi)$, it is direct computation to show that $\mathbf{y}p^*k_1(\Phi) = p^*\Phi k_1(\mathbf{y})$.

Hence, the image of the map $\beta = \mathbf{y}p^*M$ is exactly the ideal $\mathcal{I}_{\mathbb{X}}(\xi)$ and we have that:

$$\mathrm{Ann}(\mathcal{I}_{\mathbb{W}}/\mathcal{I}_{\mathbb{K}}) \simeq \mathcal{I}_{\mathbb{X}}.$$

Now we use the assumption that Z is zero-dimensional. Since the statement is local and the formation of the symmetric algebra commutes with base change (see Proposition 2.1.1), we can assume that $\mathcal{O}_{\mathbb{P}}$ and $\mathcal{O}_{\mathbb{K}}$ are Gorenstein local rings. We apply Theorem 3.1.14 to the Gorenstein scheme $\mathbb{K}$ and to the ideal sheaf $\mathcal{I}_{\mathbb{X},\mathbb{K}}$.

We denote by $\mathcal{I}_{\mathbb{W},\mathbb{K}}$ the ideal of $\mathbb{W}$ in $\mathbb{K}$. Since $\mathbb{W}$ has codimension 0 in $\mathbb{K}$ and has no embedded components, the ideals $\mathcal{I}_{\mathbb{W},\mathbb{K}}$ and $\mathcal{I}_{\mathbb{X},\mathbb{K}}$ are linked in $\mathcal{O}_{\mathbb{K}}$. This shows that $\mathcal{I}_{\mathbb{W},\mathbb{K}} = \mathrm{Ann}(\mathcal{I}_{\mathbb{X},\mathbb{K}})$. Now, since we have already $\mathcal{I}_{\mathbb{K}} \subset \mathcal{I}_{\mathbb{W}}$, the equality occurs as ideal sheaves of $\mathcal{O}_{\mathbb{P}}$ itself. Moreover we have the isomorphism $\mathrm{Ann}(\mathcal{I}_{\mathbb{X},\mathbb{K}}) \simeq (\mathcal{I}_{\mathbb{K}} : \mathcal{I}_{\mathbb{X}})$. Hence:

$$\mathcal{I}_{\mathbb{W}} = p^*\mathcal{I}_Z \simeq (\mathcal{I}_{\mathbb{K}} : \mathcal{I}_{\mathbb{X}}).$$

□

Proof of Proposition 3.2.13. As above, we can assume that $\mathcal{O}_{\mathbb{P}}$ and $\mathcal{O}_{\mathbb{K}}$ are Gorenstein local rings and we apply Theorem 3.1.14 to $\mathcal{O}_{\mathbb{K}}$. We denote again by $\mathcal{I}_{\mathbb{X},\mathbb{K}}$ the ideal of $\mathbb{X}$ in $\mathbb{K}$ and by $\mathcal{I}_{\mathbb{W},\mathbb{K}}$ the ideal of $\mathbb{W}$ in $\mathbb{K}$ (recall that $\mathbb{W} = p^*Z$).

Since $\mathcal{I}_{\mathbb{X},\mathbb{K}}$ has codimension 0 in $\mathcal{O}_{\mathbb{K}}$, we have that $(\mathcal{I}_{\mathbb{K}} : \mathcal{I}_{\mathbb{X}})$ and $\mathcal{I}_{\mathbb{X},\mathbb{K}}$ are linked. But following the notation in Lemma 3.2.14, $(\mathcal{I}_{\mathbb{K}} : \mathcal{I}_{\mathbb{X}}) \simeq \mathcal{I}_{\mathbb{W},\mathbb{K}}$.

Moreover, applying Theorem 3.1.14 in a sheafified case, $\mathbb{W}$ is Cohen-Macaulay as a pull back of Z so $\mathbb{X}$ is also Cohen-Macaulay and we have:

$$\mathcal{I}_{\mathbb{W},\mathbb{K}} \simeq \omega_{\mathbb{X}} \otimes \omega_{\mathbb{K}}^{\vee}$$

where $\omega_{\mathbb{X}}$ is the canonical sheaf of $\mathbb{X}$. We also have that:

$$\omega_{\mathbb{W}} \otimes \omega_{\mathbb{K}}^{\vee} \simeq \mathcal{I}_{\mathbb{X},\mathbb{K}} \simeq \mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}}.$$

Now, since $\mathbb{W} \simeq \mathbb{P}^n_Z$, we have $\omega_{\mathbb{W}} \simeq p^*\omega_Z \otimes \mathcal{O}_{\mathbb{P}}(-(n+1)\xi)$. Therefore, by Proposition 3.2.11:

$$\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}} \simeq \omega_{\mathbb{W}} \otimes \omega_{\mathbb{K}}^{\vee} \simeq p^*(\omega_Z \otimes \omega_X^{\vee}) \otimes \mathcal{O}_{\mathbb{P}}(-n\eta - \xi).$$

□

Denoting $\mathcal{H}_1(\mathcal{I}_Z)$ the first Koszul homology associated to

$$\Phi : \mathrm{V} \otimes \mathcal{O}_X(-\eta) \to \mathcal{O}_X,$$

as in (K3), we emphasize the following point in order to elucidate the nature of the sheaf $\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}}$.

Proposition 3.2.15. *The sheaf $\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}}$ is isomorphic to the pull-back of the first homology $\mathcal{H}_1(\mathcal{I}_Z)$ of Φ up to a shift. More precisely, we have*

$$\mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}} \simeq p^*\mathcal{H}_1(\mathcal{I}_Z) \otimes \mathcal{O}_{\mathbb{P}}(\eta - \xi).$$

Proof. To shorten the notation, we set $\mathcal{H}_1$ for $\mathcal{H}_1(\mathcal{I}_Z)$. We are going to show that

$$\mathcal{H}_1 \simeq \omega_Z \otimes \omega_X^{\vee}\big(-(n+1)\eta\big). \tag{3.2.6}$$

First, $\omega_Z \simeq \mathcal{E}\mathrm{xt}^n(\mathcal{O}_Z, \omega_X)$. Hence, we will prove (3.2.6) by showing that

$$\mathcal{O}_Z \simeq \mathcal{E}\mathrm{xt}^n(\mathcal{H}_1, \omega_X) \otimes \omega_X^{\vee}\big(-(n+1)\eta\big).$$

To this end, let:

$$\begin{tikzcd}
0 \arrow[r] & \overset{n+1}{\wedge}\mathcal{P}_1 \arrow[r, "k_n(\Phi)"] & \cdots \arrow[rr] \arrow[dr] & & \overset{2}{\wedge}\mathcal{P}_1 \arrow[rr, "k_1(\Phi)"] \arrow[dr] & & \mathcal{P}_1 \arrow[r, "\Phi"] & \mathcal{I}_Z \arrow[r] & 0 \\
& & & \mathcal{F}_2 \arrow[ur] & & \mathcal{F}_1 \subset \mathcal{E} \arrow[ur] & & &
\end{tikzcd} \tag{K4}$$

be the Koszul complex associated with $\Phi = (\phi_0 \ \dots \ \phi_n)$, where $\overset{i}{\wedge}\mathcal{P}_1 = (\wedge^i \mathrm{V}) \otimes \mathcal{O}_X(-i\eta)$. Since $\mathrm{codim}(Z, X) = \mathrm{depth}(\mathcal{I}_Z) = n$ the Koszul homology is concentrated in degree 1 and by definition $\mathcal{H}_1 = \mathcal{E}/\mathcal{F}_1$.

Applying the functor $\mathcal{H}\mathrm{om}(-, \omega_X)$ to (K4), we obtain:

$$0 \longrightarrow \mathcal{H}\mathrm{om}(\mathcal{F}_1, \omega_X) \longrightarrow \cdots \longrightarrow \mathrm{V} \otimes \omega_X(n\eta) \longrightarrow$$

$$\omega_X\big((n+1)\eta\big) \longrightarrow \mathcal{E}\mathrm{xt}^1(\mathcal{F}_{n-1}, \omega_X) \longrightarrow 0$$

and it is a computation to show that $\mathcal{E}\mathrm{xt}^1(\mathcal{F}_{n-1},\omega_X) \simeq \mathcal{E}\mathrm{xt}^{n-1}(\mathcal{F}_1,\omega_X)$.

The last point is that $\mathcal{E}\mathrm{xt}^{n-1}(\mathcal{F}_1,\omega_X) \simeq \mathcal{E}\mathrm{xt}^{n}(\mathcal{H}_1,\omega_X)$. Indeed, by the long exact sequence associated to the short exact sequence:

$$0 \to \mathcal{F}_1 \to \mathcal{E} \to \mathcal{H}_1 \to 0$$

we have the following exact sequence:

$$\mathcal{E}\mathrm{xt}^{n-1}(\mathcal{E},\omega_X) \to \mathcal{E}\mathrm{xt}^{n-1}(\mathcal{F}_1,\omega_X) \to \mathcal{E}\mathrm{xt}^{n}(\mathcal{H}_1,\omega_X) \to \mathcal{E}\mathrm{xt}^{n}(\mathcal{E},\omega_X)$$

and $\mathcal{E}\mathrm{xt}^{n-1}(\mathcal{E},\omega_X) = \mathcal{E}\mathrm{xt}^{n}(\mathcal{E},\omega_X) = 0$ since Z is locally Cohen-Macaulay.

Moreover, the last map $k_n(\Phi)$ of the Koszul complex is the transpose of the first map Φ up to signs. Thus the maps in the sequence:

$$\mathrm{V} \otimes \omega_X(n\eta) \to \omega_X\big((n+1)\eta\big) \to \mathcal{E}\mathrm{xt}^{n}(\mathcal{H}_1,\omega_X) \to 0$$

are the same as the maps in the exact sequence:

$$\mathcal{P}_1 \xrightarrow{\Phi} \mathcal{O}_X \to \mathcal{O}_Z \to 0.$$

Taking care of the twisting, this means that $\mathcal{O}_Z \otimes \omega_X\big((n+1)\eta\big) \simeq \mathcal{E}\mathrm{xt}^{n}(\mathcal{H}_1,\omega_X)$.

This implies $\mathcal{H}_1 \simeq \omega_Z \otimes \omega_X^\vee\big(-(n+1)\eta\big)$. □

Remark 3.2.16. To enlighten the construction of the sheaves $\mathcal{P}'_i$ for $i \in \{1,\dots,n+1\}$ in the following proof of Theorem 3.2.2, recall that the complex:

$$0 \longrightarrow \mathcal{P}_n \longrightarrow \dots \longrightarrow \mathcal{P}_1 \longrightarrow \mathcal{P}_0 \longrightarrow \mathcal{O}_Z \longrightarrow 0 \qquad (P_\bullet)$$

is a locally free resolution of $\mathcal{O}_Z$. Hence, applying the functor $\mathcal{H}\mathrm{om}(-,\omega_X)$ to Equation $(P_\bullet)$, a locally free resolution of ω_Z reads:

$$0 \longrightarrow \mathcal{P}_0^\vee \otimes \omega_X \longrightarrow \dots \longrightarrow \mathcal{P}_n^\vee \otimes \omega_X \longrightarrow \omega_Z \longrightarrow 0$$

(see Remark 3.1.16 for a better explanation of this fact) from which we can read a locally free resolution of $\omega_Z \otimes \omega_X^\vee$.

Proof of Theorem 3.2.2. As we saw in Lemma 3.2.8 and in the proof of Proposition 3.2.13, $\mathbb{X}$ is Cohen-Macaulay of dimension n.

Moreover, by Proposition 3.2.11 and Proposition 3.2.13, we have the following commutative diagram:

$$\begin{tikzcd}
 & & & & & & 0 \arrow[d] & \\
 & 0 \arrow[r] & \mathfrak{Q}_n \arrow[r] & \cdots \arrow[r] & \mathfrak{Q}_2 \arrow[r] & \mathfrak{Q}_1 \arrow[r] & \mathcal{I}_{\mathbb{K}} \arrow[r] \arrow[d] & 0 \\
 & & & & & & \mathcal{I}_{\mathbb{X}} \arrow[d] & \\
0 \arrow[r] & \mathcal{P}'_{n+1} \arrow[r] & \mathcal{P}'_n \arrow[r] & \cdots \arrow[r] & \mathcal{P}'_2 \arrow[r] & \mathcal{P}'_1 \arrow[r] & \mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}} \arrow[r] \arrow[d] & 0. \\
 & & & & & & 0 &
\end{tikzcd}$$

where

$$\mathfrak{Q}_i = \bigoplus_{j=0}^{i-1} \big((\overset{i+1}{\wedge} \mathrm{V}) \otimes \mathcal{O}_{\mathbb{P}}(-(j+1)\xi - (i-j)\eta) \big)$$

and

$$\mathcal{P}'_i = p^* \mathcal{P}^\vee_{n+1-i} \otimes \mathcal{O}_{\mathbb{P}}(-n\eta - \xi) \qquad \text{for } 1 \le i \le n+1.$$

Let us explain now how these resolutions patch together to give the desired resolution of $\mathcal{I}_{\mathbb{X}}$. So denote K_i the kernels (and cokernels) of the complex ($\mathfrak{Q}_\bullet$) and $\mathcal{T}_i$ as in the following diagram :

$$\begin{tikzcd}
 & & & \mathrm{K}_2 \arrow[dr] & & \mathrm{K}_1 \arrow[dr] & & 0 \arrow[d] & \\
0 \arrow[r] & \mathfrak{Q}_n \arrow[r, "d_n"'] & \cdots \arrow[ur] \arrow[r] & & \mathfrak{Q}_2 \arrow[ur] \arrow[r, "d_2"'] & \mathfrak{Q}_1 \arrow[r, "d_1"'] & \mathcal{I}_{\mathbb{K}} \arrow[r] \arrow[d] & 0 & \\
 & & & & & & \mathcal{I}_{\mathbb{X}} \arrow[d] & & \\
0 \arrow[r] & \mathcal{P}'_{n+1} \arrow[r, "\delta_{n+1}"] & \mathcal{P}'_n \arrow[r, "\delta_n"] \arrow[dr] & \cdots \arrow[r] \arrow[dr] & \mathcal{P}'_2 \arrow[r, "\delta_2"] \arrow[dr] & \mathcal{P}'_1 \arrow[r, "\delta_1"] & \mathcal{I}_{\mathbb{X}}/\mathcal{I}_{\mathbb{K}} \arrow[r] \arrow[d] & 0 & \\
 & & & \mathcal{T}_n \arrow[ur] & \mathcal{T}_2 \arrow[ur] & \mathcal{T}_1 \arrow[ur] & 0 & &
\end{tikzcd}$$

where δ_i and d_i stands for the morphisms of the associated resolutions. In order to construct the desired resolution, we first prove that δ_1 lifts to a map $\tilde{\delta}_1 : \mathcal{P}'_1 \to \mathcal{I}_X$. To do this, it suffices to prove that $\mathrm{Ext}^1\big(\mathcal{P}'_1, \mathcal{I}_{\mathbb{K}}\big) = 0$, that is $\mathrm{H}^1\big(\mathbb{P}, \mathcal{I}_{\mathbb{K}} \otimes \mathcal{P}'^\vee_1\big) = 0$. Observe that, once we show that $\tilde{\delta}_1$ exists, then the map $(d_1, \tilde{\delta}_1) : \mathfrak{Q}_1 \oplus \mathcal{P}_1 \to \mathcal{I}_{\mathbb{X}}$ is surjective. Let us now check the required vanishing. In view of ($\mathfrak{Q}_\bullet$), it suffices to show that $\mathrm{H}^i\left(\mathbb{P}, \mathfrak{Q}_i \otimes \mathcal{P}'^\vee_1\right) = 0$ for all $i \in \{1, \dots, n\}$. Proposition 3.1.6 implies these vanishings since $\mathrm{H}^i\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-j)\right) = 0$ for all $j = 0, \dots, i-1$. In the case $i = n$, we use that

$$\mathrm{H}^n\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-j)\right) \simeq \mathrm{H}^0\left(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j-n-1)\right)$$

and the fact that $j - n - 1 \le -2$.

Now consider the kernel $\mathcal{S}_1$ of the constructed morphism $\mathfrak{Q}_1 \oplus \mathcal{P}'_1 \to \mathcal{I}_{\mathbb{X}}$. We have an exact sequence

$$0 \longrightarrow \mathrm{K}_1 \longrightarrow \mathcal{S}_1 \longrightarrow \mathcal{T}_1 \longrightarrow 0.$$

Next, we construct a surjection $\mathfrak{Q}_2 \oplus \mathcal{P}'_2 \to \mathcal{S}_1$ again by lifting the map $\delta_2 : \mathcal{P}'_2 \to \mathcal{T}_1$ to $\tilde{\delta}_2 : \mathcal{P}'_2 \to \mathcal{S}_1$. This is achieved if we show that $\mathrm{Ext}^1(\mathcal{P}'_2, \mathrm{K}_1) = 0$. Again, we use a piece of ($\mathfrak{Q}_\bullet$) to show this vanishing. Indeed, we write the resolution

$$0 \longrightarrow \mathfrak{Q}_n \longrightarrow \dots \longrightarrow \mathfrak{Q}_2 \longrightarrow \mathrm{K}_1 \longrightarrow 0.$$

Applying Proposition 3.1.6 to this complex yields again the required vanishing. Therefore, we have now a surjection $(d_2, \tilde{\delta}_2) : \mathfrak{Q}_2 \oplus \mathcal{P}'_2 \to \mathcal{S}_1$. Composing with the injection of $\mathcal{S}_1$ into $\mathfrak{Q}_1 \oplus \mathcal{P}'_1$, by construction we get

$$\begin{array}{ccc} \mathfrak{Q}_2 & \begin{pmatrix} d_1 & a_1 \\ 0 & \delta_1 \end{pmatrix} & \mathfrak{Q}_1 \\ \oplus & \longrightarrow & \oplus \\ \mathcal{P}'_2 & & \mathcal{P}'_1 \end{array}$$

where a_1 is the composition of $\tilde{\delta_2}$ with the natural map $\mathcal{S}_1 \to \mathfrak{Q}_1$. Note that the map a_1 can have only constant or linear entries with respect to the y_i variables because of degree reasons. Continuing this way we construct a resolution

$$0 \longrightarrow \mathcal{P}'_{n+1} \longrightarrow \begin{matrix}\mathfrak{Q}_n\\ \oplus\\ \mathcal{P}'_n\end{matrix} \longrightarrow \ldots \longrightarrow \begin{matrix}\mathfrak{Q}_1\\ \oplus\\ \mathcal{P}'_1\end{matrix} \longrightarrow \mathcal{I}_{\mathbb{X}} \longrightarrow 0. \tag{3.2.7}$$

of $\mathcal{I}_{\mathbb{X}}$. The morphisms in the above resolution take the form

$$\begin{matrix}\mathfrak{Q}_{i+1}\\ \oplus\\ \mathcal{P}'_{i+1}\end{matrix} \xrightarrow{\begin{pmatrix} d_i & a_i \\ 0 & \delta_i\end{pmatrix}} \begin{matrix}\mathfrak{Q}_i\\ \oplus\\ \mathcal{P}'_i\end{matrix}$$

where $i \in \{1, \ldots, n\}$. These morphisms have linear or constant entries as this holds for δ_i (as d_i of course does not depend on the y_j), for a_i (by degree reasons just as for a_1) and for the d_i (by Corollary 3.2.12).

This shows Theorem 3.2.2 and Corollary 3.2.4. □

A direct application of Theorem 3.2.2 is as follows.

Corollary 3.2.17. *Under the assumption that* $\dim(Z) = 0$*, the ideal* $\mathcal{I}_{\mathbb{X}}$ *has a resolution of the following form:*

$$0 \to \mathcal{G}_{n+1} \to \mathcal{G}_n \to \ldots \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{I}_{\mathbb{X}} \to 0$$

where $\mathcal{G}_i = \bigoplus_{j=1}^{i} p^*\mathcal{T}_{ij} \otimes \mathcal{O}_{\mathbb{P}}(-j\xi)$ *when* $i \in \{1, \ldots, n\}$ *and* $\mathcal{G}_{n+1} = p^*\mathcal{T}_n \otimes \mathcal{O}_{\mathbb{P}}(-\xi)$ *for some locally free sheaves* $\mathcal{T}_{ij}$ *and* $\mathcal{T}_n$ *over* X*.*

3.2.3 Graded free resolution of the symmetric algebra

Now, we turn to the analysis of a resolution of the symmetric algebra of a homogeneous ideal of the polynomial ring $R = \mathrm{k}[x_0, \ldots, x_n]$. So let $I_Z = (\phi_0, \ldots, \phi_n) \subset R$ be an ideal generated by $n+1$ linearly independent homogeneous polynomials each one of the same degree $\eta \geq 2$. We will denote by R_Z the quotient R/I_Z and by Z the subscheme $\mathbb{V}(I_Z)$ of $\mathbb{P}^n = \mathrm{Proj}(R)$.

We will assume that $\dim(Z) = 0$ and that R_Z is a graded Cohen-Macaulay ring.

As above let:

$$0 \longrightarrow P_n \longrightarrow \ldots \longrightarrow P_2 \xrightarrow{M} P_1 \longrightarrow I_Z \longrightarrow 0 \tag{$P_\bullet$}$$

be a minimal graded free resolution of I_Z, M being the presentation matrix of I_Z and $P_1 = R(-\eta)^{n+1}$. The length of the resolution is equal to $n = \dim(\mathbb{P}^n)$ since Z is Cohen-Macaulay.

As in the previous section, let $k_1(\Phi) : \wedge^2 P_1 \to P_1$ be the second differential of the Koszul complex associated with the map $\Phi : P_1 \xrightarrow{(\phi_0 \ldots \phi_n)} R$. Put $F = \mathrm{Im}\big(k_1(\Phi)\big)$ in order to have the following diagram:

$$\begin{tikzcd}
R(-2\eta)^{\binom{n+1}{2}} \arrow[rr, "k_1(\Phi)"] \arrow[dr] & & R(-\eta)^{n+1} \arrow[r, "\Phi"] & I_Z \arrow[r] & 0. \\
& F \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

Definition 3.2.18. Set $S = R[y_0, \dots, y_n]$ and $\mathbf{y} = (y_0 \ \dots \ y_n)$. We let $I_{\mathbb{X}}$ be the ideal of S generated by the entries in the row matrix $\mathbf{y}M$ and $I_{\mathbb{K}}$ be the ideal of S generated by the entries in the row matrix $\mathbf{y}k_1(\Phi)$.

Here, as above, $F \subset E = \mathrm{Image}(M)$ so $I_{\mathbb{K}} \subset I_{\mathbb{X}}$.

Notation 3.2.19. Since S is bigraded by the variables $\mathbf{x}$ and $\mathbf{y}$, $S(-a,-b)$ stands for a shift in $\mathbf{x}$ for the left part and $\mathbf{y}$ for the right part.

As above, we denote by $\mathbb{P}$ the product $\mathbb{P}^n \times \mathbb{P}^n$ and by $p : \mathbb{P}^n \times \mathbb{P}^n \to \mathbb{P}^n$ the first projection.

To show Theorem 3.2.6, the strategy is initially the same as in the previous section, but since we are dealing with free resolutions, the resolutions of $I_{\mathbb{K}}$ and $I_{\mathbb{X}}/I_{\mathbb{K}}$ will patch together providing a resolution of $I_{\mathbb{X}}$ without further checking. We will explain afterwards how we deduce from this resolution a minimal bigraded free resolution of $I_{\mathbb{X}}$.

The Koszul hull

All the arguments of the proof of Proposition 3.2.11 remain valid in the graded homogeneous setting. So the ideal $I_{\mathbb{K}}$ has the following properties:

(i) $I_{\mathbb{K}}$ is a determinantal ideal.

Under the assumption that $\mathrm{codim}(Z, \mathbb{P}^n) = n$:

(ii) $\mathrm{codim}(\mathbb{K}, \mathbb{P}) = n$.

(iii) a graded free resolution of $I_{\mathbb{K}}$ is the Eagon-Northcott complex associated to the matrix:

$$\psi = \begin{pmatrix} \phi_0 & \dots & \phi_n \\ y_0 & \dots & y_n \end{pmatrix}.$$

Hence, the following complex is a bigraded free resolution of $I_{\mathbb{K}}$:

$$0 \longrightarrow Q_n \longrightarrow \dots \longrightarrow Q_2 \longrightarrow Q_1 \longrightarrow I_{\mathbb{K}} \longrightarrow 0 \qquad (Q_\bullet)$$

where $Q_i = \bigoplus_{j=0}^{i-1} Q_{i,j}$ and

$$Q_{i,j} = S\big(-(i-j)\eta, -j-1\big)^{\binom{n+1}{i+1}} \qquad \text{for } 1 \le i \le n \text{ and } 0 \le j \le i-1.$$

(iv) The scheme $\mathbb{K}$ is Gorenstein and the canonical module $\omega_{S_{\mathbb{K}}}$ of $\mathbb{K}$ verifies:

$$\omega_{S_{\mathbb{K}}} \simeq S\big(n(\eta-1)-1, -n\big).$$

Identification of the quotient $I_{\mathbb{X}}/I_{\mathbb{K}}$

We denote by ω_{R_Z} the canonical module of Z. All the arguments of Proposition 3.2.11 and Theorem 3.1.14 apply in the graded case since R_Z is a graded Cohen-Macaulay ring of depth n. Hence we have that:

$$I_{\mathbb{X}}/I_{\mathbb{K}} \simeq \omega_{R_Z} \otimes S(n(1-\eta)+1,-1) \text{ as } S\text{-modules.}$$

Recall that $(P_\bullet)$ is a minimal graded free resolution of I_Z. Put

$$P'_i = P^\vee_{n+1-i} \otimes S\big(-n\eta,-1\big) \quad \text{ for } i \in \{1,\dots,n+1\}.$$

Then the complex:

$$0 \longrightarrow P'_{n+1} \longrightarrow \begin{matrix} Q_n \\ \oplus \\ P'_n \end{matrix} \longrightarrow \dots \longrightarrow \begin{matrix} Q_2 \\ \oplus \\ P'_2 \end{matrix} \longrightarrow \begin{matrix} Q_1 \\ \oplus \\ P'_1 \end{matrix} \longrightarrow I_{\mathbb{X}} \longrightarrow 0 \tag{R2'}$$

is a bigraded free resolution of $I_{\mathbb{X}}$.

Homotopy of complexes

We turn now to the problem of extracting a minimal bigraded free resolution of $I_{\mathbb{X}}$ from (R2'). In order to do so, we show first the following result.

Proposition 3.2.20. *There is a canonical isomorphism*

$$p_*\mathcal{O}_{\mathbb{X}}(\xi) \simeq \mathcal{I}_Z$$

where $\mathcal{O}_{\mathbb{X}}(\xi)$ and $\mathcal{I}_Z$ are the sheafification of respectively $S(0,1)$ and I_Z.

We emphasize that this is not completely straight forward since $\mathbb{X}$ is the Proj of $\mathcal{I}_Z$ which is not locally free (see Stack project, 26.21. Projective bundles, example 26.21.2).

Proof. Since $\mathcal{O}_{\mathbb{P}}(\xi)$ is the relative ample line bundle of the projective bundle $\mathbb{P} = \mathbb{P}\big(\mathcal{O}_X(-\eta)^{n+1}\big)$, we have:

$$\mathrm{R}^k p_*\mathcal{O}_{\mathbb{P}}(l\eta - j\xi) = \begin{cases} 0 & \text{for } l>0 \text{ and } j \le 0, \\ 0 & \text{for } j \in \{1,\dots,k-1\} \text{ and any } l, \\ \mathcal{O}_X(l\eta) & \text{for } k=0 \text{ and } j=0, \\ \mathcal{O}_X^{n+1}\big((l-1)\eta\big) & \text{for } k=0 \text{ and } j=-1. \end{cases}$$

Therefore, applying p_* to the resolution (R1) and chasing cohomology we get $\mathrm{R}^1 p_*\mathcal{I}_{\mathbb{X}}(\xi) = 0$.

Recall that we denote by $\mathcal{E}$ the kernel of $\Phi : \mathcal{O}_X(-\eta)^{n+1} \to \mathcal{I}_Z$ and that $\mathcal{I}_{\mathbb{X}}(\xi)$ is the image of the map $p^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}}(\xi)$. Let $\mathcal{H}$ be the kernel of this surjection and write the exact sequence:

$$0 \to \mathcal{H} \to p^*\mathcal{E} \to \mathcal{I}_{\mathbb{X}}(\xi) \to 0.$$

Since $p_*p^*\mathcal{E} \simeq \mathcal{E}$ and $\mathrm{R}^1p_*p^*\mathcal{E} = 0$, applying p_* to this exact sequence, we get:

$$0 \to p_*\mathcal{H} \to \mathcal{E} \to p_*\mathcal{I}_{\mathbb{X}}(\xi) \to \mathrm{R}^1p_*\mathcal{H} \to 0. \tag{a}$$

Also, since we proved that $\mathrm{R}^1p_*\mathcal{I}_{\mathbb{X}}(\xi) = 0$, applying p_* to the canonical exact sequence

$$0 \to \mathcal{I}_{\mathbb{X}}(\xi) \to \mathcal{O}_{\mathbb{P}}(\xi) \to \mathcal{O}_{\mathbb{X}}(\xi) \to 0$$

we get

$$0 \to p_*\mathcal{I}_{\mathbb{X}}(\xi) \to \mathcal{O}_X(-\eta)^{n+1} \to p_*\mathcal{O}_{\mathbb{X}}(\xi) \to 0. \tag{b}$$

The exact sequences (a) and (b) fit into the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & & \\
 & p_*\mathcal{H} \arrow[d] & 0 \arrow[d] & & \\
 & \mathcal{E} \arrow[r, "\simeq"] \arrow[d] & \mathcal{E} \arrow[d] & & \\
0 \arrow[r] & p_*\mathcal{I}_{\mathbb{X}}(\xi) \arrow[r] \arrow[d] & \mathcal{O}_X(-\eta)^{n+1} \arrow[r] \arrow[d] & p_*\mathcal{O}_{\mathbb{X}}(\xi) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{R}^1p_*\mathcal{H} \arrow[r] \arrow[d] & \mathcal{I}_Z \arrow[r] \arrow[d] & p_*\mathcal{O}_{\mathbb{X}}(\xi) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where (a) is the left column, (b) is the central row and the map $\mathcal{I}_Z \to p_*\mathcal{O}_{\mathbb{X}}(\xi)$ in the bottom row is the canonical morphism associated to the projectivization of $\mathcal{I}_Z$. This morphism is an isomorphism over $X \backslash Z$ and therefore $\mathcal{I}_Z \to p_*\mathcal{O}_{\mathbb{X}}(\xi)$ is injective because $\mathcal{I}_Z$ is torsion free. Hence $p_*\mathcal{H} \simeq 0 \simeq \mathrm{R}^1p_*\mathcal{H}$ and $p_*\mathcal{O}_{\mathbb{X}}(\xi) \simeq \mathcal{I}_Z$. $\square$

Proof of Theorem 3.2.6. We work as in the previous proposition. Applying p_* to the resolution of $\mathcal{O}_{\mathbb{X}}(\xi)$ given by (R1) and considering the associated R-modules of global sections, we obtain the following graded free resolution of I_Z:

$$0 \longrightarrow P_0^\vee\big(-(n+1)\eta\big) \longrightarrow \begin{matrix} R\big(-(n+1)\eta\big) \\ \oplus \\ P_0^\vee\big(-(n+1)\eta\big) \end{matrix} \longrightarrow \cdots \longrightarrow$$

$$\begin{matrix} R(-2\eta)^{\binom{n+1}{2}} \\ \oplus \\ P_n^\vee(-(n+1)\eta) \end{matrix} \longrightarrow R(-\eta)^{n+1} \longrightarrow I_Z \longrightarrow 0.$$

This resolution is homotopic to the minimal free resolution $(P_\bullet)$ of I_Z. Therefore, the truncated complex $(P_{\geq 1})$ of $(P_\bullet)$ is homotopic as a S-complex to:

$$0 \longrightarrow P'_{n-1} \longrightarrow \begin{matrix} Q_{n,0} \\ \oplus \\ P'_n \end{matrix} \longrightarrow \dots \longrightarrow \begin{matrix} Q_{1,0} \\ \oplus \\ P'_1 \end{matrix}.$$

Hence, (R2') is homotopic to:

$$0 \longrightarrow Q''_n \longrightarrow \begin{matrix} Q''_{n-1} \\ \oplus \\ P''_{n-1} \end{matrix} \longrightarrow \begin{matrix} Q''_{n-2} \\ \oplus \\ P''_{n-2} \end{matrix} \longrightarrow \dots \longrightarrow \begin{matrix} Q''_2 \\ \oplus \\ P''_2 \end{matrix} \longrightarrow P''_1 \longrightarrow I_{\mathbb{X}} \longrightarrow 0 \qquad \text{(R2)}$$

where

$$Q''_i = \bigoplus_{j=1}^{n} Q_{i,j}, \qquad Q_{i,j} = S\big(-(i-j)\eta, -j-1\big)^{\binom{n+1}{i+1}}, \qquad P''_i = P_{i+1} \otimes S(\eta, -1).$$

The complex (R2) is thus a bigraded free resolution of $I_{\mathbb{X}}$.

To finish the proof of Theorem 3.2.6, it remains to show that (R2) is minimal. This follows from the minimality of $(P_\bullet)$ and the fact that, if $i \neq i'$, there is no bigraded homogeneous piece of the same degree among Q''_i and $Q''_{i'}$ or P''_j for any $j \in \{1, \dots, n-1\}$. $\square$

Part II

Application to the study of rational maps

Chapter 4

n-to-n-tic maps

As we saw in Chapter 2, given a rational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ defined by $n+1$ homogeneous polynomials $\phi_0, \ldots, \phi_n \in R = \mathrm{k}[x_0, \cdots, x_n]$ and denoting $\mathcal{I}_Z = (\phi_0, \ldots, \phi_n)$ for the ideal sheaf of the base locus Z of Φ, we can study two schemes related to $\mathcal{I}_Z$. First, there is the graph Γ of Φ which is the **Proj** of the Rees algebra of $\mathcal{I}_Z$. By definition, the projective degrees of Φ are defined as the coefficients of the decomposition of Γ in cycle classes in $\mathbb{P}^n \times \mathbb{P}^n$. Hence, we can read from Γ whether Φ is birational. Second, there is the projectivization $\mathbb{X}$ of the ideal of $\mathcal{I}_Z$ which is the **Proj** of $\mathrm{Sym}(\mathcal{I}_Z)$. Actually $\mathbb{X}$ contains Γ as an irreducible component and we have moreover the following commutative diagram:

$$\begin{tikzcd}
 & \mathbb{P}^n \times \mathbb{P}^n \arrow[d, hook] \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\
 & \mathbb{X} \arrow[d, hook] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & \Gamma \arrow[dll, "\sigma_1"'] \arrow[drr, "\sigma_2"] & \\
\mathbb{P}^n \arrow[rr, dashed, "\Phi"'] & & \mathbb{P}^n
\end{tikzcd}$$

where p_1 (resp. p_2) is the projection over the first factor $\mathbb{P}^n$ (resp. second factor $\mathbb{P}^n$).

Notation. In the following, and with the notation in the previous diagram, we denote by h_1 (resp. h_2) the class of a pull back of a hyperplane of $\mathbb{P}^n$ by p_1 (resp. p_2).

Given a vector bundle $\mathcal{G}$ over $\mathbb{P}^n \times \mathbb{P}^n$ and two integers i and j, we denote also by $\mathcal{G}(i,j)$ the twisted vector bundle $\mathcal{G}(ih_1 + jh_2)$.

The equations of $\mathbb{X}$ and Γ in $\mathbb{P}^n \times \mathbb{P}^n$ are closely related to a free presentation of $\mathcal{I}_Z$. Namely, let

$$\begin{tikzcd}
\bigoplus_{i\geq 1} \mathcal{O}_{\mathbb{P}^n}(-i)^{b_i} \arrow[rr, "M"] \arrow[dr] & & \mathcal{O}_{\mathbb{P}^n}^{n+1} \arrow[r, "\begin{pmatrix}\phi_0 & \dots & \phi_n\end{pmatrix}"] & \mathcal{I}_Z(\delta) \arrow[r] & 0 \\
& \mathcal{E} \arrow[ur] \arrow[dr] & & & \\
0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

be a locally free presentation of $\mathcal{I}_Z$ and where $\mathcal{E}$ is the image of M in $\mathcal{O}_{\mathbb{P}^n}^{n+1}$. Writing $y_0, \dots, y_n$ for the variables of $\mathbb{P}^n$, the equations of $\mathbb{X}$ in $\mathbb{P}^n \times \mathbb{P}^n$ are the entries of the row matrix $\begin{pmatrix}y_0 & \dots & y_n\end{pmatrix} M$ and we can compute the equations of Γ from these equations.

Definition 4.0.1. We call the sheaf $\mathcal{E}$, *the sheaf of relations* of $\mathcal{I}_Z$.

Hence, if $\mathcal{E}$ is locally free, $\mathbb{X}$ is the zero locus of a global section of the vector bundle $p_1^*\mathcal{E}^\vee(h_2)$ over $\mathbb{P}^n \times \mathbb{P}^n$. Moreover, if $\mathcal{E}$ is split as a sum of n line bundles $\mathcal{O}_{\mathbb{P}^n}(-a_i)$, $p_1^*\mathcal{E}^\vee(h_2)$ is equal to the sum $\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(a_i h_1 + h_2)$ and $\mathbb{X}$ is thus the intersection of n divisors in $\mathbb{P}^n \times \mathbb{P}^n$. Hence, when $\mathbb{X}$ has pure dimension n, it is a complete intersection in $\mathbb{P}^n \times \mathbb{P}^n$ and this last case makes particularly easy to compute the naive multidegree of Φ. As we will see in Definition 4.1.1 the naive multidegree of Φ is defined in the same way as the multidegree of Φ is defined, that is the multidegree $(d_0, \dots, d_n)$ of Φ is the decomposition of Γ in cycle classes in $\mathrm{CH}(\mathbb{P}^n \times \mathbb{P}^n)$ and the naive multidegree $(\mathfrak{d}_0, \dots, \mathfrak{d}_n)$ of Φ is the decomposition of $\mathbb{X}$ in cycle classes in $\mathrm{CH}(\mathbb{P}^n \times \mathbb{P}^n)$.

In the present chapter and Chapter 5, we focus on two situations for which $\mathcal{E}$ is locally free. The first one is to assume that $\mathcal{E}$ is split. The second one will be to work on the base variety $X = \mathbb{P}^2$. Let focus on the first case.

By the Hilbert-Burch theorem [Eis95, 20.15], in the case that Z has the expected codimension 2, the fact that $\mathcal{E}$ is split is equivalent to the fact that $\mathcal{I}_Z$ is the ideal of maximal minors of M. Let us illustrate this with the following situation extracted from [DH17, Example 4.17].

Example 4.0.2. Let Φ be the rational map defined by the 3×3-minors of the matrix $M = \begin{pmatrix} -x_1 & x_0 & -x_1^2 + x_0x_3 \\ x_0 & x_1 & x_0^2 - x_1x_2 \\ 0 & x_2 & x_0x_1 - x_1x_3 \\ 0 & x_3 & -x_0x_1 + x_0x_2 \end{pmatrix}$. Hence the base ideal sheaf $\mathcal{I}_Z$ has the following locally free presentation:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3_1}(-2) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^3_1}^4 \xrightarrow{\begin{pmatrix}\phi_0 & \dots & \phi_3\end{pmatrix}} \mathcal{I}_Z(4) \longrightarrow 0$$

where $\mathcal{O}_{\mathbb{P}^3_1}$ is the structure sheaf of $\mathbb{P}^3_1$.

Hence, the ideal $I_{\mathbb{X}}$ of $\mathbb{X}$ in $\mathbb{P}^3_1 \times \mathbb{P}^3_2$ is generated by the entries of the row matrix $\begin{pmatrix}y_0 & \dots & y_3\end{pmatrix} M$, and since via a Macaulay2 computation we know that $\mathbb{X}$ has dimension n, we have that $\mathbb{X}$ is the complete intersection of two hypersurfaces of

bidegree $(1,1)$ and one hypersurface of bidedegree $(2,1)$. Hence the decomposition of $\mathbb{X}$ in cycle classes is

$$(h_1+h_2)^2(2h_1+h_2) = 2h_1^3 + 5h_1^2h_2 + 4h_1h_2^2 + h_2^3$$

where $h_1 = c_1\big(p_1^*\big(\mathcal{O}_{\mathbb{P}^n}(1)\big)\big)$ and $h_2 = c_1\big(p_2^*\big(\mathcal{O}_{\mathbb{P}^n_2}(1)\big)\big)$. Thus $(\mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3) = (2,5,4,1)$.

Now, we focus on the decomposition of $\mathbb{X}$ into its irreducible components. As we explained in Proposition 2.2.13, the possible torsion components are fibres over components of $\mathbb{V}\big(\mathrm{Fitt}_i(\mathcal{I}_Z)\big)$ for $i \in \{1,2\}$. Here $\mathbb{V}\big(\mathrm{Fitt}_2(\mathcal{I}_Z)\big)$ is empty but $\mathrm{Fitt}_1(\mathcal{I}_Z)$ is supported over $\mathbb{V}(x_0,x_1)$ in $\mathbb{P}^3_1$. Actually, it is a computation, for example by computing the *primary decomposition* of $\mathbb{X}$ with Macaulay2, to show that the ideal of the torsion component $\mathbb{T}$ in $\mathbb{P}^3 \times \mathbb{P}^3$ is $\mathcal{I}_{\mathbb{T}} = (x_0, x_1, y_2x_2 + y_3x_3)$.

To sum up $\mathbb{X}$ decomposes as the union of two components Γ and $\mathbb{T}$ each one of dimension 3. These two components have to be taken in account for the computation of the naive projective degrees of $\mathbb{X}$. In this case, $\mathbb{T}$ is a complete intersection so its multidegree $(1,1,0,0)$ is easy to determine. Hence, the multidegree of Γ is $(2-1, 5-1, 4, 1) = (1,4,4,1)$. To explain this subtraction, we refer to the geometric interpretation of the naive multidegree or multidegree. For example, the nth naive projective degree, is the coefficient of h_2^3 in the decomposition of $\mathbb{X}$. But since $\mathbb{T}$ intersect h_2^3 once, Γ must intersect h_2^3 once too.

By the previous analysis, we want to emphasize that the data of determinantal maps with given multidegree is equivalent to the data of the ideal of minors of the presentation matrix M.

There are two main motivations for this chapter. First, as we mentioned in Theorem 4, the multidegree of a Cremona map verifies the Cremona inequalities but it is not known if given a sequence $(d_0, \ldots, d_n)$ verifying Cremona inequalities, there exists a Cremona map $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Hence, the construction we propose is already a way to partially answer, even with n big, Problem A concerning the Cremona inequalities.

The second motivation concerns more precisely particular birational maps $\Phi : \mathbb{P}^3_1 \dashrightarrow \mathbb{P}^3_2$ whose inverses Φ^{-1} have the same algebraic degree than Φ. For simplicity we call these maps *n-to-n-tics* by analogy with the more common denomination cubo-cubic and quarto-quartic in degree 3 and 4. Recall from Subsection 1.3.3 that the second multidegree d_2 of Φ is the algebraic degree of Φ^{-1}. Hence n-to-n-tic maps are the maps with multidegree $(1,n,n,1)$. Since quadro-quadrics $(1,2,2,1)$ rational maps of $\mathbb{P}^3$ cannot be defined by the maximal minors of a (4×3)-matrix of non constant entries, we did not focus on this case. Let us emphasize however that these quadro-quadric maps have been studied and classified via the XJC-correspondance (see [PR16]) in [PR13] and [PR14].

4.1 Expected degrees and naive projective degrees

With the previous notation, we let $\mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be rational map.

Definition 4.1.1. We define the *naive multidegree* $(\mathfrak{d}_0, \dots, \mathfrak{d}_n)$ *of* Φ as the multidegree of $\mathbb{X}$ of the base ideal sheaf $\mathcal{I}$ of Φ in $\mathbb{P}^n \times \mathbb{P}^n$, i.e.

$$[\mathbb{X}] = \sum_{i=0}^{n} \mathfrak{d}_i h_1^i h_2^{n-i}$$

For $i \in \{0, \dots, n\}$, we call $\mathfrak{d}_i$ the ith naive projective degree.

Let us illustrate this definition with the nth projective degree.

Example 4.1.2. Let $\Phi : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be a rational map. In this case the base locus Z of Φ has codimension 2 so, by 2.2.19, the possible torsion components of $\mathbb{X}$ have dimension 2 in $\mathbb{P}^2 \times \mathbb{P}^2$ and $\mathbb{X}$ has pure dimension 2.

Since the graph Γ of Φ (resp. $\mathbb{X}$) is defined with two morphisms $\sigma_1 : \Gamma \to \mathbb{P}^2$ and $\sigma_2 : \Gamma \to \mathbb{P}^2$ (resp. $\pi_1 : \mathbb{X} \to \mathbb{P}^2$ and $\pi_2 : \mathbb{X} \to \mathbb{P}^2$) which are the restriction to Γ (resp. to $\mathbb{X}$) of the two projections $p_1 : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$ and $p_2 : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^2$, we summarize the situation with the following commutative diagram:

$$\begin{tikzcd}
 & \mathbb{P}^2 \times \mathbb{P}^2 \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
 & \mathbb{X} \arrow[u, hook] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & \Gamma \arrow[u, hook] \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\
\mathbb{P}^2 \arrow[rr, dashed, "\Phi"'] & & \mathbb{P}^2.
\end{tikzcd}$$

In this case, $c_2\big((\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0,1)^2)|_{\mathbb{X}}\big)$ and $c_2\big((\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0,1)^2)|_{\Gamma}\big)$ are both 0-cycles of $\mathbb{P}^2 \times \mathbb{P}^2$ and the nth projective degree of Φ is the topological degree:

$$d_t(\Phi) = \deg(c_2\big((\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0,1)^2)\big)|_{\Gamma})$$

and the 2nd naive projective degree of Φ is:

$$\mathfrak{d}_2 = \deg(c_2\big((\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(0,1)^2)|_{\mathbb{X}}\big)).$$

Let $n \geq 1$ and let $\mathcal{I}$ be an ideal sheaf generated by $n+1$ global sections of $\mathcal{O}_{\mathbb{P}^n}(\delta)$, identified with homogeneous polynomials in $n+1$ variables of degree δ. Assume that the sheaf of relations $\mathcal{E}$ of $\mathcal{I}$ is locally free, i.e. a locally free resolution of $\mathcal{I}$ reads

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow \mathcal{I}(\delta) \longrightarrow 0.$$

Concerning this naive multidegree we have the following result.

Proposition 4.1.3. *The kth naive projective degree of Φ is equal to the degree of the $n+1-k^{th}$ Chern class $c_{n+1-k}(\mathcal{E}^\vee)$ of $\mathcal{E}^\vee$ (since $\mathrm{CH}^{n+1-k}(\mathbb{P}^n) \simeq \mathbb{Z}$).*

Proof. The kth projective degree of Φ is the degree of the support of the cokernel W^{n+1-k} of a general morphism $\mathcal{O}_{\mathbb{P}^n}^k \to \mathcal{I}(\delta)$. So we consider the following commutative diagram:

$$\begin{tikzcd}
 & & \mathcal{O}_{\mathbb{P}^n}^k \arrow[r, phantom, "="] \arrow[d] & \mathcal{O}_{\mathbb{P}^n}^k \arrow[d] & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[dr] & \mathcal{O}_{\mathbb{P}^n}^{n+1} \arrow[r, "(\phi_0 \ \dots \ \phi_n)"] \arrow[d] & \mathcal{I}(\delta) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_{\mathbb{P}^n}^{n+1-k} \arrow[r] & \mathcal{O}_{W^{n+1-k}} \arrow[r] & 0
\end{tikzcd}$$

and we focus on the sequence $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^n}^{n+1-k} \to \mathcal{O}_{W^{n+1-k}} \to 0$. If W^{n+1-k} has the expected codimension $n+1-k$, then, by Proposition 1.2.6, the class $[W^{n+1-k}]$ is equal to the $n+1-k^{\text{th}}$ Chern class $c_{n+1-k}(\mathcal{E}^\vee)$ of $\mathcal{E}^\vee$. □

Proposition 4.1.4. *Now suppose that the sheaf of relations $\mathcal{E}$ of $\mathcal{I}$ is split and equal to $\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^n}(-a_i)$ for some integers $a_i \geq 1$.*

Then the naive multidegree of the rational map Φ associated to $\mathcal{I}$ is

$$(\mathfrak{d}_n, \dots, \mathfrak{d}_0) = (\mathfrak{s}_n(a_1, \dots, a_n), \dots, \mathfrak{s}_0(a_1, \dots, a_n))$$

where $\mathfrak{s}_i(a_1, \dots, a_n)$ is the ith elementary function in $a_1, \dots, a_n$.

Proof. This follows from applying Proposition 4.1.3 and computing the Chern classes of $\mathcal{E}$. □

Now, with the notation of the introduction to Chapter 4, recall that the base ideal of a rational map Φ is of linear type if $\mathbb{X} = \Gamma$. This implies that

$$(d_n, \dots, d_0) = (\mathfrak{d}_n, \dots, \mathfrak{d}_0).$$

Proposition 4.1.5. *[DHS12, Corollary 2.6] Let $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a rational map whose base ideal $\mathcal{I}_Z$ has a split sheaf of relation and that $Z = \mathbb{V}(\mathcal{I}_Z)$ has codimension 2.*

Assume that $\mathcal{I}$ is of linear type, then Φ is birational if and only if the generators $\phi_0, \dots, \phi_n$ of $\mathcal{I}$ are the maximal minors of a $n \times n$ matrix M with only linear entries.

Proof. Since $\mathcal{E}$ is split, put $\mathcal{E} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^n}(-a_i)$ and write

$$\bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^n}(-a_i) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^n}^{n+1} \xrightarrow{\begin{pmatrix}\phi_0 & \dots & \phi_n\end{pmatrix}} \mathcal{I}(\delta) \longrightarrow 0$$

a locally free presentation of $\mathcal{I}$ (where $\delta = \sum_i a_i i$). As we explained in the introduction, by the Hilbert-Burch theorem [Eis95, 20.15], this is equivalent to the fact that the $\phi_0, \dots, \phi_n$ are the $n \times n$-minors of M (because Z has codimension 2).

Now assuming that $\mathcal{I}_Z$ is of linear type, of course if M has only linear entries i.e. $a_i = 1$ for all i, $d_n = \mathfrak{d}_n = \mathfrak{s}_n(1, \dots, 1) = 1$ so Φ is birational.

Conversely, if Φ is birational, $d_n = \mathfrak{d}_n = \mathfrak{s}_n(a_1, \dots, a_n) = 1$ which implies that $a_i = 1$ for all i since the a_i are integers. So M has only linear entries. □

We use the following notion of generality:

Definition 4.1.6. We say that a statement holds for a *general matrix* M if there exists a Zariski dense subset U of $\mathrm{H}^0(p_1^*(\mathcal{E}^\vee)(h_2))$ such that the statement holds for all p_1^*M in U. If p_1^*M lies in the complement of countably many Zariski closed subsets of $\mathrm{H}^0(p_1^*(\mathcal{E}^\vee)(h_2))$, we say that M is *very general*.

4.2 Quarto-quartics

A determinantal rational map $\Phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ of algebraic degree 4 is the data of its presentation (4×3)-matrix M necessarily with two columns of linear entries and one column with quadratic entries, i.e. a locally free resolution of the base ideal sheaf of $\mathcal{I}_Z$ reads:

$$0 \longrightarrow \mathcal{E} = \mathcal{O}_{\mathbb{P}^3_1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3_1}(-2) \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^3_1} \xrightarrow{\begin{pmatrix}\phi_0 & \cdots & \phi_3\end{pmatrix}} \mathcal{I}_Z(4) \longrightarrow 0.$$

Proposition 4.2.1. *If p_1^*M is general in $\mathrm{H}^0(p_1^*(\mathcal{E}^\vee)(h_2))$, then the multidegree and naive multidegree of Φ are the same and are equal to $(2, 5, 4, 1)$.*

Proof. First, this is a theorem of Bertini [Har77, 7.9.1] ($p_1^*(\mathcal{E}^\vee)(h_2)$ being very ample since all the a_i are strictly positive) that a section of $\mathrm{H}^0(p_1^*(\mathcal{E}^\vee)(h_2))$ is smooth and irreducible so the projectivization $\mathbb{X}$ of $\mathcal{I}_Z$ is equal to the blow-up of $\mathcal{I}_Z$. In other words, $\mathcal{I}_Z$ is of linear type so the multidegree and the naive multidegree of Φ coincide.

Now the symmetric functions in $1, 1, 2$ give the multidegree $(2, 5, 4, 1)$. □

As we saw, in order to create a torsion component which decreases the 3rd and 2nd projective degree by one, we can make the matrix M have rank 1 over a curve in $\mathbb{P}^3_1$. Since some of the entries of M are linear this curve is necessarily a line, for instance the line $\mathbb{V}(x_0, x_1)$. We construct a quarto-quartic map as follows. Let

$$M_1 = \begin{pmatrix} a_0^1x_0 + a_1^1x_1 + a_2^1x_2 + a_3^1x_3 \\ a_0^2x_0 + a_1^2x_1 + a_2^2x_2 + a_3^2x_3 \\ a_0^3x_0 + a_1^3x_1 + a_2^3x_2 + a_3^3x_3 \\ a_0^4x_0 + a_1^4x_1 + a_2^4x_2 + a_3^4x_3 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} b_0^1x_0 + b_1^1x_1 \\ b_0^2x_0 + b_1^2x_1 \\ b_0^3x_0 + b_1^3x_1 \\ b_0^4x_0 + b_1^4x_1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} c^1_{2000}x_0^2 + c^1_{1100}x_0x_1 + c^1_{1010}x_0x_2 + c^1_{1001}x_0x_3 + c^1_{0200}x_1^2 + c^1_{0110}x_1x_2 + c^1_{0101}x_1x_3 \\ c^2_{2000}x_0^2 + c^2_{1100}x_0x_1 + c^2_{1010}x_0x_2 + c^2_{1001}x_0x_3 + c^2_{0200}x_1^2 + c^2_{0110}x_1x_2 + c^2_{0101}x_1x_3 \\ c^3_{2000}x_0^2 + c^3_{1100}x_0x_1 + c^3_{1010}x_0x_2 + c^3_{1001}x_0x_3 + c^3_{0200}x_1^2 + c^3_{0110}x_1x_2 + c^3_{0101}x_1x_3 \\ c^4_{2000}x_0^2 + c^4_{1100}x_0x_1 + c^4_{1010}x_0x_2 + c^4_{1001}x_0x_3 + c^4_{0200}x_1^2 + c^4_{0110}x_1x_2 + c^4_{0101}x_1x_3 \end{pmatrix}$$

be the respective first, second and third columns of M where the coefficients a_j^i, b_j^i and c^i_{klmn} are in k, and assume that the collection of those polynomials is general.

Proposition 4.2.2. *Given that M is general among the conditions we imposed, i.e. p_1^*M is general in*

$$\mathrm{H}^0\Big(p_1^*\big(\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{I}_{\mathbb{V}(x_0,x_1)}(1) \oplus \mathcal{I}_{\mathbb{V}(x_0,x_1)}(2)\big)(h_2)\Big),$$

the determinantal map of the 3×3 *minors of* M *has multidegree* $(1, 4, 4, 1)$.

Proof. To define M, put $L = \mathbb{V}(x_0, x_1)$ and let $L_1 = p_1^*(L)$ in $\mathbb{P}^3 \times \mathbb{P}^3$. Our definition of the generality of M is the same as considering s_1 general in $\mathrm{H}^0\big(\mathcal{O}_{\mathbb{P}^3 \times \mathbb{P}^3}(1,1)\big)$, s_2 general in $\mathrm{H}^0\big(\mathcal{I}_{L_1}(1,1)\big)$, s_3 general in $\mathrm{H}^0\big(\mathcal{I}_{L_1}(2,1)\big)$ and considering that $\mathbb{X}$ is the intersection $\mathbb{V}(s_1) \cap \mathbb{V}(s_2) \cap \mathbb{V}(s_3)$. But this subvariety is smooth outside L_1 by a theorem of Bertini [Har77, 7.9.1] so, by analysing the Fitting ideals of M, for instance $\mathrm{Fitt}_1(M)$ is supported over $\mathbb{V}(x_0, x_1)$ and the other Fitting ideals are zero, it has two components. One is necessarily the blow-up Γ of $\mathcal{I}$. We denote by $\mathbb{T}$ the other one supported over L_1. This latter component $\mathbb{T}$ is reduced. Indeed, by Bertini's theorem, we can show this property of being reduced by studying the intersection $\mathbb{V}(s_2) \cap \mathbb{V}(s_3)$ for s_2 general in $\mathrm{H}^0\big(\mathcal{I}_{L_1}(1,1)\big)$ and s_3 general in $\mathrm{H}^0\big(\mathcal{I}_{L_1}(2,1)\big)$ and showing that it is reduced. Since being reduced is an open property, it suffices to show the reduction for a specific choice of s_2 and s_3. So let

$$\mathbb{V}(s_2) = (H_1 \times \mathbb{P}^3) \cup (\mathbb{P}^3 \times H_2)$$

where H_1 is a hyperplane of the first factor $\mathbb{P}^3$ containing L and

$$\mathbb{V}(s_3) = (C_1 \times \mathbb{P}^3) \cup (\mathbb{P}^3 \times H_2')$$

where C_1 is a quadric surface in the first factor $\mathbb{P}^3$ containing L. Then, in particular since the intersection of H_1 and C_1 in $\mathbb{P}^3$ is equal to the (reduced) union of L with another line L', we have

$$\mathbb{V}(s_2) \cap \mathbb{V}(s_3) = (L \times \mathbb{P}^3) \cup (L' \times \mathbb{P}^3) \cup (H_1 \times H_2') \cup (C_1 \times H_2) \cup \big(\mathbb{P}^3 \times (H_2 \cap H_2')\big)$$

which is reduced. Hence $\mathbb{T}$ is reduced. So, since $\mathbb{T}$ coincides set-theoretically with $\mathbb{P}(\mathcal{I}(4)|_L)$, we have $\mathbb{T} = \mathbb{P}(\mathcal{I}(4)|_L)$.

Actually $\mathcal{I}(4)|_L \simeq \mathcal{O}_L^2 \oplus \mathcal{O}_L(1)$ by restriction of the presentation matrix M of $\mathcal{I}$ over L where the columns of M vanish. Hence $\mathbb{P}\big(\mathcal{I}(4)|_L\big)$ is a $\mathbb{P}^2$-fibration over L so that the coefficients of $h_1 h_2^2$ and h_2^3 are respectively 1 and 1. To check the second coefficient, note that $\mathbb{P}(\mathcal{I}(4)|_L) = \mathbb{P}\big(\mathcal{O}_L^2 \oplus O_L(1)\big)$ and that the map given by the relatively ample line bundle is the blow-up of a line in $\mathbb{P}^3$. So the cycle $h_1^3 \mathbb{T}$ is given by the inverse image of a general point of $\mathbb{P}^3$ under this blow up, and is thus a single point.

For the first coefficient, restricting $\mathcal{I}(4)|_L$ to a general plane P we get $\mathcal{O}_x^3$ where $x = L \cap P$. So the cycle $\mathbb{P}(\mathcal{I}(4)|_L) h_1$ is the plane $\mathbb{P}(O_x^3)$, which maps to a plane in $\mathbb{P}^3$ and hence cuts a general line along a single point.

To sum up, $\mathbb{T}$ is the intersection of L_1 with a general divisor of class $(1,1)$ in $\mathbb{P}^3 \times \mathbb{P}^3$, so $L_1 = h_1^2$ and $\mathbb{T} = h_1^2(h_1 + h_2)$. So the multidegree of $\mathbb{T}$ is $(1, 1, 0, 0)$ and the multidegree of Γ is necessarily $(1, 4, 4, 1)$. □

In the article [DH17], J.Déserti and F.Han provide a description of the base locus of such a general quarto-quartic map.

Proposition 4.2.3. *Let* $\Phi : \mathbb{P}_1^3 \dashrightarrow \mathbb{P}_2^3$ *be a quarto-quartic map whose presentation matrix* M *verifies that* $p_1^* M$ *is general in*

$$\mathrm{H}^0\Big(p_1^*\big(\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{I}_{\mathbb{V}(x_0,x_1)}(1) \oplus \mathcal{I}_{\mathbb{V}(x_0,x_1)}(2)\big)(h_2)\Big).$$

Then the base locus Z of Φ is the union of the line $\mathrm{L} = \mathbb{V}(\mathrm{x}_0, \mathrm{x}_1)$ of multiplicity 3 and a smooth irreducible curve C of genus 5 and degree 8. Moreover L is 5-secant to C.

Let us emphasize that the total degree 11 of the union of the line L and C is coherent with the fact that the curve defined by the ideal $\mathcal{I}_3(M)$ of 3×3-minors of a matrix M such that p_1^*M is general in $\mathrm{H}^0(p_1^*(\mathcal{O}_{\mathbb{P}^3}(1)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(2))(h_2))$ is a smooth curve of degree 11. Indeed, consider M as a matrix with entry in the polynomial ring $R = \mathrm{k}[x_0, x_1, x_2, x_3]$ and consider the ideal $I_3(M)$ of 3×3-minors of M. By the genericity of M, the Eagon-Northcott complex associated to M provided a minimal free resolution of $R/I_3(M)$. Namely,

$$0 \;\to\; \mathcal{O}_{\mathbb{P}^3_1}(-5)^2 \oplus \mathcal{O}_{\mathbb{P}^3_1}(-6) \xrightarrow{M} \mathcal{R}(-4)^4 \;\to\; R \;\to\; R/I_3(M) \;\to\; 0 \tag{4.2.1}$$

is a minimal free resolution of $R/I_3(M)$. Now, as it is explained in [Sch03, 3.2 Free resolutions, page 46], we can compute the Hilbert polynomial of $I_3(M)$ via this resolution, it is equal to $HP(I_3(M), T) = 11T - 13$ so the degree of the curve defined by $I_3(M)$ in $\mathbb{P}^3$ is equal to 11.

Proposition 4.2.4. *[DH17, Remark 4.6] The inverse of a determinantal quarto-quartic map as in Proposition 4.2.3 is a determinantal quarto-quartic map.*

Remark 4.2.5. Let us explain how computationally (empirically) we recover this fact. Let $\Phi : \mathbb{P}^3_1 \dashrightarrow \mathbb{P}^3_2$ be a determinantal quarto-quartic general map for which we denote by

$$0 \;\longrightarrow\; \mathcal{O}_{\mathbb{P}^3_1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3_1}(-2) \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^3_1} \xrightarrow{\begin{pmatrix}\phi_0 & \dots & \phi_3\end{pmatrix}} \mathcal{I}_Z(4) \;\longrightarrow\; 0.$$

a presentation of $\mathcal{I}_Z$. We can assume without lost of generality that the line supporting the ideal sheaf $\mathrm{Fitt}_0(\mathcal{I}_Z)$ is the line $\mathbb{V}(x_0, x_1)$ and that M is given as in the proof of Proposition 4.2.3. The ideal $\mathcal{I}_{\mathbb{X}}$ of the projectivization of $\mathbb{X}$ in $\mathbb{P}^3_1 \times \mathbb{P}^3_2$ is generated by two sections of bidegree $(1,1)$ and one section of bidegree $(2,1)$ (the entries of the row matrix $\begin{pmatrix}y_0 & y_1 & y_2 & y_3\end{pmatrix} M$).

Then it is a computation to show that the ideal $\mathcal{I}_\Gamma$ of the graph Γ of Φ, that is the saturation $[\mathcal{I}_{\mathbb{X}} : (x_0, x_1)^\infty]$ of $\mathcal{I}_{\mathbb{X}}$ by (x_0, x_1) (where we consider (x_0, x_1) as an ideal sheaf over $\mathbb{P}^3_1 \times \mathbb{P}^3_2$) is generated by two sections of bi-degree $(1,1)$, one section of bi-degree $(2,1)$ and one section of bi-degree $(1,2)$. Now the projectivization $\mathcal{I}_{\mathbb{X}'}$ of the projectivization $\mathbb{X}'$ of the base ideal $\mathcal{I}_{Z'}$ of Φ^{-1} should be generated by the two sections of bidegree $(1,1)$ and the section of bidegree $(1,2)$ generating $\mathcal{I}_\Gamma$. Indeed, $\mathbb{X}'$ contains the graph Γ of Φ (because it is also the graph of Φ^{-1}) as an irreducible component and is generated by at least three sections of bi-degree $(1,*)$ because Γ has codimension 3. In this case, these three sections of bi-degree $(1,*)$ are indeed the two sections of bidegree $(1,1)$ and the section of bidegree $(1,2)$ generating $\mathcal{I}_\Gamma$. Since the ideal of the projectivization $\mathcal{I}_{\mathbb{X}'}$ of the projectivization $\mathbb{X}'$ of the base ideal $\mathcal{I}_{Z'}$ of Φ^{-1} is generated by two sections of bidegree $(1,1)$ and one section of bidegree $(1,2)$, the presentation matrix of $\mathcal{I}_{Z'}$ has two columns with linear entries and one column of quadratic entries and Φ^{-1} is thus a determinantal (quarto-quartic) map. See Chapter 7 for the details of the last argument.

One way to complete our proof that $\mathbb{X}'$ is indeed generated by the two sections of bidegree $(1,1)$ and the section of bidegree $(1,2)$ generating $\mathcal{I}_\Gamma$ would be to show that $\Gamma = \mathbb{X} \cap \mathbb{X}'$ but, even if we could verify this fact in all our examples, we could not provide of a complete proof of it.

Remark 4.2.6. We emphasize that given the line $\mathbb{V}(x_0, x_1)$, we can consider the matrix

$$M = \begin{pmatrix} \lambda_{01}^1 & l_{01}^1 & q_{0123}^1 \\ \lambda_{01}^2 & l_{01}^2 & q_{0123}^2 \\ \lambda_{01}^3 & l_{01}^3 & q_{0123}^3 \\ \lambda_{01}^4 & l_{01}^4 & q_{0123}^4 \end{pmatrix}$$

where the entries l_{01}^i and λ_{01}^i are linear polynomials in the variables x_0, x_1 and the entries q_{0123}^i are quadratic polynomials in the variables x_0, x_1, x_2, x_3. Let Φ be the determinantal map of the 3×3 minors of M. In this case, the torsion component $\mathbb{T}$ of $\mathbb{X}$ has ideal $\mathcal{I}_\mathbb{T} = (x_0, x_1, \sum_{i=0}^{3} y_i q_{0123}^{i+1})$ and hence its decomposition in $\mathrm{CH}(\mathbb{P}_1^3 \times \mathbb{P}_2^3)$ is $h_1^2(2h_1 + h_2)$ and its multidegree is thus $(2,1,0,0)$ so the multidegree of the graph Γ is $(0,4,4,1)$ and Φ is not dominant. This is why we only consider the case that M has one of its column with linear entries and the columns of quadratic entries which vanish over the line $\mathbb{V}(x_0, x_1)$.

Now, we can make an enumeration of the parameters of the determinantal quarto-quartics. As we see, up to a choice of coordinate, a determinantal quarto-quartic is the data of a matrix M such that

$$M_1 = \begin{pmatrix} a_0^1 x_0 + a_1^1 x_1 + a_2^1 x_2 + a_3^1 x_3 \\ a_0^2 x_0 + a_1^2 x_1 + a_2^2 x_2 + a_3^2 x_3 \\ a_0^3 x_0 + a_1^3 x_1 + a_2^3 x_2 + a_3^3 x_3 \\ a_0^4 x_0 + a_1^4 x_1 + a_2^4 x_2 + a_3^4 x_3 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} b_0^1 x_0 + b_1^1 x_1 \\ b_0^2 x_0 + b_1^2 x_1 \\ b_0^3 x_0 + b_1^3 x_1 \\ b_0^4 x_0 + b_1^4 x_1 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} c_{2000}^1 x_0^2 + c_{1100}^1 x_0 x_1 + c_{1010}^1 x_0 x_2 + c_{1001}^1 x_0 x_3 + c_{0200}^1 x_1^2 + c_{0110}^1 x_1 x_2 + c_{0101}^1 x_1 x_3 \\ c_{2000}^2 x_0^2 + c_{1100}^2 x_0 x_1 + c_{1010}^2 x_0 x_2 + c_{1001}^2 x_0 x_3 + c_{0200}^2 x_1^2 + c_{0110}^2 x_1 x_2 + c_{0101}^2 x_1 x_3 \\ c_{2000}^3 x_0^2 + c_{1100}^3 x_0 x_1 + c_{1010}^3 x_0 x_2 + c_{1001}^3 x_0 x_3 + c_{0200}^3 x_1^2 + c_{0110}^3 x_1 x_2 + c_{0101}^3 x_1 x_3 \\ c_{2000}^4 x_0^2 + c_{1100}^4 x_0 x_1 + c_{1010}^4 x_0 x_2 + c_{1001}^4 x_0 x_3 + c_{0200}^4 x_1^2 + c_{0110}^4 x_1 x_2 + c_{0101}^4 x_1 x_3 \end{pmatrix}$$

are the respective first, second and third columns of M. This a total of 52 parameters. However, different choices of parameters can produce the same map Φ. This is the case for example if we compose a matrix $G \in \mathrm{Gl}_4(\mathrm{k})$ with M. This should give a way to recover the following result

Proposition 4.2.7. *[DH17, Proposition 4.8] The family of determinantal quarto-quartic has dimension* 46 *in the Cremona group* $\mathrm{Bir}(\mathbb{P}^3, \mathbb{P}^3)$.

However, we did not have time to push that far our results.

Determinantal quinto-quartic interlude

With our point of view of enumerating the torsion components of the projectivization of the base ideal $\mathcal{I}_Z$ of a determinantal quartic map, we can consider the case

of quinto-quartics, that is maps of $\mathbb{P}^3$ of multidegree $(1,5,4,1)$. The resolution of $\mathcal{I}_Z$ still reads

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3_1}(-2) \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^3_1} \xrightarrow{\begin{pmatrix}\phi_0 & \cdots & \phi_3\end{pmatrix}} \mathcal{I}_Z(4) \longrightarrow 0.$$

and Φ still has naive multidegree $(\mathfrak{d}_3, \mathfrak{d}_2, \mathfrak{d}_1, \mathfrak{d}_0) = (2,5,4,1)$. Hence, here we are looking for a torsion component of dimension 3 influencing $\mathfrak{d}_3$ but not $\mathfrak{d}_2$, that is torsion component with cohomological decomposition h_1^3 in $\mathrm{CH}(\mathbb{P}^3_1 \times \mathbb{P}^3_2)$. Those possible torsion components are thus located above a closed point of $\mathbb{P}^3$. So let choose the point $\mathbb{V}(x_0, x_1, x_2)$. Since it is of codimension 3 in $\mathbb{P}^3$, it has to be the support of $\mathrm{Fitt}_2(\mathcal{I}_Z)$ i.e. support of the ideal of 1×1 minors of M.

For instance, let

$$M = \begin{pmatrix} \lambda^1_{012} & l^1_{012} & q^1_{012} \\ \lambda^2_{012} & l^2_{012} & q^2_{012} \\ \lambda^3_{012} & l^3_{012} & q^3_{012} \\ \lambda^4_{012} & l^4_{012} & q^4_{012} \end{pmatrix}$$

where the entries l^i_{012} and λ^i_{012} are linear polynomials in the variables x_0, x_1, x_2 and the entries q^i_{012} are quadratic polynomials whose monomials are all divisible by the variables x_0, x_1, x_2 but else general in those conditions. The map of 3×3 minors of M has then multidegree $(1,5,4,1)$.

Proposition 4.2.8. *Let z be a point in $\mathbb{P}^3$ and let $z_1 = p_1^* z$. Now let M be a matrix such that $p_1^* M$ is very general in*

$$\mathrm{H}^0\big(\mathcal{I}_{z_1}(1,1)^2 \oplus \mathcal{I}_{z_1}(2,1)\big)\big).$$

Then the determinantal map Φ associated to the 3×3-minors of M has multidegree $(1,5,4,1)$ i.e. is a quinto-quartic.

Proof. In the same way as in the proof of Proposition 4.2.2, the projectivization $\mathbb{X}$ of the base ideal sheaf $\mathcal{I}$ of Φ has two reduced components. One is the blow-up Γ of Φ, the other is $\mathbb{V}(z_1)$ which has multidegree $(1,0,0,0)$. Hence, since the multidegree of $\mathbb{X}$ is $(2,5,4,1)$, the multidegree of Γ is $(1,5,4,1)$ which is thus the multidegree of Φ. □

Computationally we observe that the base locus Z of a general determinantal quinto-quartic is a curve of degree 11 and arithmetic genus 14, singular at the support of the torsion component. Its inverse is a determinantal quintic whose base locus Z' is the union of two singular curve of degree 9, one of arithmetic genus 7, the other of arithmetic genus 8. We give the computation with Macaulay2. We make the computation over $\mathbb{Z}/5\mathbb{Z}$ because it is much simpler. In order to impose the general conditions on polynomials whose monomials are divisible by x_0, x_1, x_2 we give a weight to each variable x_0, x_1, x_2, x_3 at first.

```
i1 : k = ZZ/5

o1 = k
```

```
o1 : QuotientRing

i2 : R = k[x_0,x_1,x_2,x_3,Degrees=>{{1,0,0,0},{0,1,0,0},
{0,0,1,0},{0,0,0,1}}]

o2 = R

o2 : PolynomialRing
```

We then define the polynomials in the matrix M.

```
i3 : Q1 = random({2,0,0,0},R)+random({1,1,0,0},R)+
random({1,0,1,0},R)+random({1,0,0,1},R)+random({0,2,0,0},R)+
random({0,1,1,0},R)+random({0,1,0,1},R)+random({0,0,2,0},R)+
random({0,0,1,1},R)

      2            2                    2
o3 = x  - 2x x  - x  - x x  + 2x x  - 2x  + x x
      0     0 1    1    0 2     1 2     2    0 3

o3 : R

i4 : Q2 = random({2,0,0,0},R)+random({1,1,0,0},R)+
random({1,0,1,0},R)+random({1,0,0,1},R)+random({0,2,0,0},R)+
random({0,1,1,0},R)+random({0,1,0,1},R)+random({0,0,2,0},R)+
random({0,0,1,1},R)


        2            2
o4 = - x  - 2x x  + x  + 2x x  + 2x x  + x x
        0     0 1    1     1 2     0 3    2 3

o4 : R

i5 : Q3 = random({2,0,0,0},R)+random({1,1,0,0},R)+
random({1,0,1,0},R)+random({1,0,0,1},R)+random({0,2,0,0},R)+
random({0,1,1,0},R)+random({0,1,0,1},R)+random({0,0,2,0},R)+
random({0,0,1,1},R)


        2     2                    2
o5 = - x  + 2x  - x x  + 2x x  + x  - 2x x  + 2x x
        0     1    0 2     1 2    2     0 3     2 3

o5 : R
```

```
i6 : Q4 = random({2,0,0,0},R)+random({1,1,0,0},R)+
random({1,0,1,0},R)+random({1,0,0,1},R)+random({0,2,0,0},R)+
random({0,1,1,0},R)+random({0,1,0,1},R)+random({0,0,2,0},R)+
random({0,0,1,1},R)

      2            2                    2
o6 = x  - x x  - 2x  - x x  - 2x x  - x  + x x
      0    0 1     1    0 2     1 2    2    2 3

o6 : R

i7 : L1 = random({1,0,0,0},R)+random({0,1,0,0},R)+
random({0,0,1,0},R)

o7 = -x
       0

o7 : R

i8 : L2 = random({1,0,0,0},R)+random({0,1,0,0},R)+
random({0,0,1,0},R)

o8 = 2x  + 2x  - x
       0     1    2

o8 : R

i9 : L3 = random({1,0,0,0},R)+random({0,1,0,0},R)+
random({0,0,1,0},R)

o9 = 2x  + 2x
       0     1

o9 : R

i10 : L4 = random({1,0,0,0},R)+random({0,1,0,0},R)+
random({0,0,1,0},R)

o10 = 2x  - 2x
        1     2

o10 : R

i11 : K1 = random({1,0,0,0},R)+random({0,1,0,0},R)+
random({0,0,1,0},R)

o11 = x  + x  + x
       0    1    2
```

```
o11 : R

i12 : K2 = random({1,0,0,0},R)+random({0,1,0,0},R)+
random({0,0,1,0},R)

o12 = -x
        1

o12 : R

i13 : K3 = random({1,0,0,0},R)+random({0,1,0,0},R)+
random({0,0,1,0},R)

o13 = 2x  - 2x  + x
        0     1    2

o13 : R

i14 : K4 = random({1,0,0,0},R)+random({0,1,0,0},R)+
random({0,0,1,0},R)

o14 = 2x  - 2x  + 2x
        0     1     2

o14 : R

i15 : M = matrix{{Q1,Q2,Q3,Q4},{L1,L2,L3,L4},{K1,K2,K3,K4}}

o15 = | x_0^2-2x_0x_1-x_1^2-x_0x_2+2x_1x_2-2x_2^2+x_0x_3
      | -x_0
      | x_0+x_1+x_2
      ----------------------------------------------------
      -x_0^2-2x_0x_1+x_1^2+2x_1x_2+2x_0x_3+x_2x_3
      2x_0+2x_1-x_2
      -x_1
      ----------------------------------------------------
      -x_0^2+2x_1^2-x_0x_2+2x_1x_2+x_2^2-2x_0x_3+2x_2x_3
      2x_0+2x_1
      2x_0-2x_1+x_2
      ----------------------------------------------------
      x_0^2-x_0x_1-2x_1^2-x_0x_2-2x_1x_2-x_2^2+x_2x_3 |
      2x_1-2x_2                                       |
      2x_0-2x_1+2x_2                                  |

              3       4
o15 : Matrix R  <--- R
```

```
i16 :
      T = k[x_0,x_1,x_2,x_3]

o16 = T

o16 : PolynomialRing

i17 : M = sub(M,T);

i18 : I = minors(3,M);

i19 :
      degree I,genus I

o19 = (11, 14)

o19 : Sequence

i20 : time primaryDecomposition I
     -- used 0.131974 seconds
```

We do not print the output corresponding to the command primaryDecomposition of $\mathcal{I}$ but the result is that $\mathbb{V}(I)$ is irreducible because there is one element in the resulting list.

```
i21 :
time radical (primaryDecomposition ideal singularLocus I)_0
     -- used 1.13423 seconds

o21 = ideal (x , x , x )
              2   1   0

o21 : Ideal of T

i23 : loadPackage "Cremona";

o23 = Cremona

o23 : Package

i24 : psi = toMap(minors(3,M));

o24 : RingMap T <--- T

i28 : projectiveDegrees psi

o28 = {1, 4, 5, 1}
```

```
o28 : List

i30 : phi = inverseMap psi;

o30 : RingMap T <--- T

i31 :
idPhi = ideal( phi(x_0) , phi(x_1) , phi(x_2) , phi(x_3));

i32 : betti res idPhi

             0 1 2
o32 = total: 1 4 3
          0: 1 . .
          1: . . .
          2: . . .
          3: . . .
          4: . 4 2
          5: . . .
          6: . . 1

o32 : BettiTally
```

Remark 4.2.9. Let us remark that we could determine the multidegree of Φ without using the Cremona package. Indeed, the torsion component $\mathbb{T}$ is supported over $\mathbb{V}(x_0, x_1, x_2)$ so we can saturate $\mathbb{X}$ in order to recover the blow-up of $\mathcal{I}$. This is the content of the following code. We start the previous code at the line `o21`.

```
i22 : S = T[y_0,y_1,y_2,y_3]

o22 = S

o22 : PolynomialRing

i23 : J = ideal( matrix{{y_0,y_1,y_2,y_3}}*sub(transpose M,S));

o23 : Ideal of S

i24 : Graph = saturate(J,sub(ideal(x_0,x_1,x_2),S));

o24 : Ideal of S
```

Via the generator of the ideal `Graph` we can then compute the kth projective degree d_k of Φ by intersecting Γ with the plane of $\mathbb{P}^3 \times \mathbb{P}^3$ of class $h_1^{3-k} h_2^k$. We provide one command for the computation of d_3 (the other projective degree can be computed in a similar way).

```
i25 : H3 = ideal (matrix{{y_0,y_1,y_2,y_3}}*
random(S^{4:{1,1}},S^{3:{1,1}}));
```

```
o25 : Ideal of S

i26 : F3 = Graph+H3;

o26 : Ideal of S

i27 : PDF3 = primaryDecomposition F3


o27  = {ideal (x  - x , x  - x , x  + x , y , y  - 2y , y  + y ),
               2    3   1    3   0    3   3   1     2   0    2
      -------------------------------------------------------------
                                                                 3
 ideal (y , y , y , y ), ideal (x , x , x , y , y  - 2y , y  + y , y )}
         3   2   1   0           2   1   0   3   1     2   0    2   2

o27 : List

i28 : degree PDF3_0

o28 = 1
```

traducing the fact that $d_3 = 1$ and that Φ is birational.

4.3 Determinantal quinto-quintics

We turn now to determinantal quinto-quintics. We emphasize that there is two partitions of the number 5 in three positive integers, namely $5 = 1 + 1 + 3$ and $5 = 1 + 2 + 2$. Hence the resolution of the base ideal sheaf $\mathcal{I}_Z$ of a determinantal quinto-quintic is one of the followings:

$$(1) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3_1}(-3) \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^3_1} \longrightarrow \mathcal{I}_Z(5) \longrightarrow 0$$

$$(2) \quad 0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_1}(-1) \oplus \mathcal{O}_{\mathbb{P}^3_1}(-2)^2 \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^3_1} \longrightarrow \mathcal{I}_Z(5) \longrightarrow 0$$

We propose constructions in both cases.

(1) We focus first on the case that the base ideal sheaf $\mathcal{I}_Z$ of a determinantal quinto-quintic $\Phi : \mathbb{P}^3_1 \dashrightarrow \mathbb{P}^3_2$ has resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3_1}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3_1}(-3) \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^3_1} \longrightarrow \mathcal{I}_Z(5) \longrightarrow 0$$

We denote by $\mathbb{X}$ the projectivization of $\mathcal{I}_Z$. It is the complete intersection of two divisors of bidegree $(1, 1)$ and one divisor of bidegree $(3, 1)$. Hence its

decomposition in $\mathrm{CH}(\mathbb{P}^3_1 \times \mathbb{P}^3_2)$ is:

$$[\mathbb{X}] = (h_1 + h_2)^2(3h_1 + h_2) = 3h_1^3 + 7h_1^2h_2 + 5h_1h_2^2 + h_2^3.$$

Thus the naive multidegree $(\mathfrak{d}_0, \mathfrak{d}_1, \mathfrak{d}_2, \mathfrak{d}_3)$ of Φ is equal to $(3, 7, 5, 1)$ i.e. if M has a general collection of entries, Φ has multidegree $(3, 7, 5, 1)$ (this follows also from Proposition 4.1.4). Hence if Φ has multidegree $(1, 5, 5, 1)$, $\mathbb{X}$ must present torsion components whose multidegree adds up to $(2, 2, 0, 0)$ which is to say, torsion components whose total class is $2h_1^3 + 2h_1^2h2$.

We propose two constructions for this multidegree.

(a) The first solution is to construct two torsion components each one of mutidegree $(1, 1, 0, 0)$. So we choose two lines in $\mathbb{P}^3_1$, for instance $\mathbb{V}(x_0, x_1)$ and $\mathbb{V}(x_2, x_3)$ and we let

$$M = \begin{pmatrix} \lambda^1_{01} & l^1_{23} & c^1 \\ \lambda^2_{01} & l^2_{23} & c^2 \\ \lambda^3_{01} & l^3_{23} & c^3 \\ \lambda^4_{01} & l^4_{23} & c^4 \end{pmatrix}$$

where for $i \in \{1, \ldots, 4\}$, the entries λ^i_{01} (resp. l^i_{23}) are linear homogeneous polynomial in the variables x_0, x_1 (resp. x_2, x_3) and c^i is a homogeneous polynomial of degree 3 with all its monomials divisible by x_0x_2 or x_0x_3 or x_1x_2 or x_1x_3.

Proposition 4.3.1. *Let M be the previous matrix and assume that it is general among the conditions we imposed. Then the determinantal map of the 3×3 minors of Φ is a quinto-quintic. We call such a map a* quinto-quintic of type (a).

Proof. Since the entries of M are general among the conditions we imposed, we have that $\mathrm{Fitt}_1(\mathcal{I}_Z)$ is supported over the union $\mathbb{V}(x_0, x_1) \cup \mathbb{V}(x_2, x_3)$. Hence the two torsion components over the lines $\mathbb{V}(x_0, x_1)$ and $\mathbb{V}(x_2, x_3)$ have dimension 3 and decomposition $h_1^2(h_1 + h_2) = h_1^3 + h_1^2h_2$ in $\mathrm{CH}(\mathbb{P}^3_1 \times \mathbb{P}^3_2)$, we refer to Remark 4.2.6 for the details of this argument. Hence, since there is no other torsion components, the component Γ of the graph has multidegree $(3-2, 7-2, 5, 1) = (1, 5, 5, 1)$ i.e. Φ is a quinto-quintic. □

Example 4.3.2. An example of quinto-quintic of type (a) is the the determinantal map of the 3×3 minors of the following presentation matrix:

$$M = \begin{pmatrix} x_0 & x_2 + x_3 & x_0^2x_2 + x_0x_1x_2 + x_0x_3^2 \\ 3x_0 + x_1 & x_2 + 2x_3 & x_1^2x_3 + x_1x_2x_3 \\ x_0 + x_1 & x_2 & x_1x_2^2 + x_0x_1x_3 \\ x_0 + 2x_1 & x_3 & x_0^2x_3 + x_1x_3^2 \end{pmatrix}$$

The base locus Z of Φ has degree 18 and is the union of the two lines $\mathbb{V}(x_0, x_1)$ and $\mathbb{V}(x_2, x_3)$ and an irreducible smooth curve of degree 12 and genus 9. Moreover this latter curve is 8-secant to $\mathbb{V}(x_0, x_1)$ and 8-secant $\mathbb{V}(x_2, x_3)$.

Remark 4.3.3. In all the examples of quinto-quintic of type (a) we have considered, the inverse was also of type (a) but we could not establish this fact in all generality.

(b) We turn now to the construction of a single torsion component of multidegree $(2,2,0,0)$. As we said in the introduction, for the moment, we only control the reduced structure of $\mathbb{X}$ which, concerning the location of the torsion components, is computed by the fitting ideal of the base ideal sheaf $\mathcal{I}_Z$. Hence, our strategy is to construct a torsion component with big enough multidegree supported over a line, for example $\mathbb{V}(x_0, x_1)$.

For instance, consider the matrix

$$M = \begin{pmatrix} l^1_{0123} & l^1_{01} & c^1_{01} \\ l^2_{0123} & l^2_{01} & c^2_{01} \\ l^3_{0123} & l^3_{01} & c^3_{01} \\ l^4_{0123} & l^4_{01} & c^4_{01} \end{pmatrix}$$

where for all $i \in \{1, \dots, 4\}$ l^i_{0123} is a linear homogeneous polynomial in the variables x_0, x_1, x_2, x_3, l^i_{01} is a linear homogeneous polynomial in the variables x_0, x_1 and c^i_{01} is a cubic homogeneous polynomial whose monomials are divisible by x_0^2 or x_0x_1 or x_1^2. If the entries of M are general under those conditions, our conjecture is that the associated determinantal map is a quinto-quintic. We call *determinantal quinto-quintic of type (b)*. An example of such a quinto-quintic is as follows, the computation showing this result and the properties of the base locus was made with Macaulay2.

Proposition 4.3.4. *Let*

$$M = \begin{pmatrix} x_0 & x_0 + x_2 + x_3 & x_0^3 + x_0^2x_1 + x_1^3 + 2x_0^2x_2 \\ x_1 & x_1 + x_3 & 2x_1^3 + x_0^2x_2 + x_0x_1x_3 \\ x_0 + x_1 & x_2 + x_3 & 2x_0^2x_2 + x_0x_1x_2 + x_0^2x_3 \\ x_0 & x_2 + x_3 & x_1^3 + 2x_0x_1x_2 + x_1^2x_2 + x_1^2x_3 \end{pmatrix}.$$

Then the determinantal map associated to 3×3 minors of M is a quinto-quintic (of type b).

Its base locus ideal I_Z has degree 18 and is the union of a smooth irreducible curve of genus 8 and degree 11 and the line $\mathbb{V}(x_0, x_1)$. Moreover the curve is 8 secant to $\mathbb{V}(x_0, x_1)$. The inverse of Φ is also a quinto-quintic of type (b).

(2) We turn now to the case that the base ideal sheaf $\mathcal{I}_Z$ of a determinantal quinto-quintic $\Phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^3$ has resolution

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1) \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^2 \xrightarrow{M} \mathcal{O}^4_{\mathbb{P}^3} \longrightarrow \mathcal{I}_Z(5) \longrightarrow 0.$$

If M is general in $\mathrm{H}^0(\mathcal{O}_{\mathbb{P}^3}(1) \oplus \mathcal{O}_{\mathbb{P}^3}(2)^2)$, the decomposition of $\mathbb{X}$ in $\mathrm{CH}(\mathbb{P}^3 \times \mathbb{P}^3)$ is:

$$[\mathbb{X}] = (h_1 + h_2)(2h_1 + h_2)^2 = 4h_1^3 + 8h_1^2h_2 + 5h_1h_2^2 + h_2^3$$

where $h_1 = c_1\big(p_1^*\big(\mathcal{O}_{\mathbb{P}^n}(1)\big)\big)$ and $h_2 = c_1\big(p_2^*\big(\mathcal{O}_{\mathbb{P}^n_2}(1)\big)\big)$ so

$$(\mathfrak{d}_3, \mathfrak{d}_2, \mathfrak{d}_1, \mathfrak{d}_0) = (4, 8, 5, 1).$$

In this case, our strategy is to construct one torsion component with dimension strictly bigger than n, for instance, by providing the vanishing of M over the line $\mathbb{V}(x_0, x_1)$. Here, $\mathbb{X}$ is no more a complete intersection so the naive multidegree of the associated map is not $(4, 8, 5, 1)$ anymore.

Hence, consider the matrix

$$M = \begin{pmatrix} l_{01}^1 & k_{01}^1 & q_{01}^1 \\ l_{01}^2 & k_{01}^2 & q_{01}^2 \\ l_{01}^3 & k_{01}^3 & q_{01}^3 \\ l_{01}^4 & k_{01}^4 & q_{01}^4 \end{pmatrix}$$

where for all $i \in \{1, \dots, 4\}$ l_{01}^i is a linear homogeneous polynomial in the variables x_0, x_1 and k_{01}^i and q_{01}^i are quadric homogeneous polynomials whose monomial are divisible by x_0 or x_1. We observe that if the entries of M are general among those conditions, then the determinantal map of 3×3 minors of M is a quinto-quintic whose inverse is a determinantal quinto-quintic whose projectivization of base locus $\mathbb{X}'$ has also a single torsion component supported over a line. An example of such a quinto-quintic is as follows, the computation showing this result and the properties of the base locus was made with Macaulay2.

Proposition 4.3.5. *Let*

$$M = \begin{pmatrix} 2x_0 + x_1 & 2x_1^2 + x_0x_3 & x_0^2 + x_0x_3 \\ x_0 & x_1x_2 + x_1x_3 & x_0^2 + x_0x_1 + x_1x_2 \\ x_0 + x_1 & x_0^2 + x_0x_2 & x_0x_1 + x_1^2 \\ x_0 & x_0^2 + x_1^2 & x_0^2 + x_1x_2 + x_1x_3 \end{pmatrix}.$$

Then the determinantal map associated to the 3×3 minors of M is a quinto-quintic.

Its base locus ideal I_Z has degree 17 and is the union of a smooth irreducible curve of genus 8 and degree 11 and the line $\mathbb{V}(x_0, x_1)$. Moreover the curve is 8-secant to $\mathbb{V}(x_0, x_1)$. The inverse of Φ is also a quinto-quintic of of the same type.

4.4 Higher degree and higher dimension

To finish this section, we propose now a summary of our investigation until now. Instead of writing explicitly the conditions under which the entries of the matrix M has to be taken general, we write an example of matrix whose associated determinantal map should represent the general properties of the family.

4.4.1 Some families of $\mathbb{P}^3$

Multidegree	Example	Base locus Z
$(1,4,4,1)$		
(112)	$\begin{pmatrix} -x_1 & x_0 & -x_1^2+x_0x_3 \\ x_0 & x_1 & x_0^2-x_1x_2 \\ 0 & x_2 & x_0x_1-x_1x_3 \\ 0 & x_3 & -x_0x_1+x_0x_2 \end{pmatrix}$	Z has degree 11 and is the union of a smooth irreducible curve $\mathcal{C}$ of degree 8 and genus 5 and a line l. The curve $\mathcal{C}$ is 5-secant to l.
$(1,5,4,1)$		
(112)	$\begin{pmatrix} x_2 & x_1 & x_0^2+x_0x_3 \\ x_0+x_2 & x_0+x_2 & x_0^2+x_1x_3 \\ x_0+x_1 & 0 & x_1x_3+x_2x_3 \\ 0 & x_2 & x_0^2+x_1x_2 \end{pmatrix}$	Z is an irreducible curve $\mathcal{C}$ of degree 11 and arithmetic genus 14 singular in one point.
$(1,5,5,1)$		
(113(a))	$\begin{pmatrix} x_0 & x_2+x_3 & x_0^2x_2+x_0x_1x_2+x_0x_3^2 \\ 3x_0+x_1 & x_2+2x_3 & x_1^2x_3+x_1x_2x_3 \\ x_0+x_1 & x_2 & x_1x_2^2+x_0x_1x_3 \\ x_0+2x_1 & x_3 & x_0^2x_3+x_1x_3^2 \end{pmatrix}$	Z has degree 18 and is the union of a smooth irreducible curve $\mathcal{C}$ of degree 12 and genus 9 and two lines l_1 and l_2. The curve $\mathcal{C}$ is 8-secant to l_1 and 8-secant to l_2.
(113(b))	$\begin{pmatrix} x_0 & x_2+x_3 & x_0^2x_2+x_0x_1x_2+x_0x_3^2 \\ 3x_0+x_1 & x_2+2x_3 & x_1^2x_3+x_1x_2x_3 \\ x_0+x_1 & x_2 & x_1x_2^2+x_0x_1x_3 \\ x_0+2x_1 & x_3 & x_0^2x_3+x_1x_3^2 \end{pmatrix}$	Z has degree 18 and is the union of a smooth irreducible curve $\mathcal{C}$ of degree 11 and genus 8 and one line l. The curve $\mathcal{C}$ is 8-secant to l.
(122)	$\begin{pmatrix} 2x_0+x_1 & 2x_1^2+x_0x_3 & x_0^2+x_0x_3 \\ x_0 & x_1x_2+x_1x_3 & x_0^2+x_0x_1+x_1x_2 \\ x_0+x_1 & x_0^2+x_0x_2 & x_0x_1+x_1^2 \\ x_0 & x_0^2+x_1^2 & x_0^2+x_1x_2+x_1x_3 \end{pmatrix}$	Z has degree 17 and is the union of a smooth irreducible curve $\mathcal{C}$ of degree 11 and genus 8 and one line l. The curve $\mathcal{C}$ is 8-secant to l_1 and 8-secant to l_2.

We emphasize that in the example of quinto-quintic of type (122), the projectivization $\mathbb{X}$ has a torsion component of dimension 4 above the line $\mathbb{V}(x_0, x_1)$.

Multidegree	Example	Base locus Z
$(1,7,5,1)$		
(113)	$\begin{pmatrix} x_0+x_2 & x_2 & x_0^3+x_1^3+x_0x_2^2+x_2^3+x_0x_1x_3 \\ x_0+x_1 & x_1+x_2 & x_0^2x_1+x_1^3+x_0x_2^2+x_2^2x_3 \\ x_2 & x_0 & x_0^2x_2+x_0x_1x_2+x_1^2x_2+x_1x_2x_3 \\ x_0+x_1 & x_2 & x_0x_1^2+x_0^2x_3+x_1^2x_3+x_1x_2x_3 \end{pmatrix}$	Z is a singular irreducible curve of degree 18 and arithmetic genus 39.
$(1,6,6,1)$		
(114(a))	$\begin{pmatrix} x_0 & x_2 & x_1^2x_2^2+x_0^2x_2x_3+x_0^2x_3^2 \\ x_1 & x_2 & x_0^2x_1x_2+x_0^2x_2^2+x_0^3x_3+x_0x_1x_2x_3 \\ x_0+x_1 & x_2-x_3 & x_0^2x_1x_2+x_1^3x_2+x_0^2x_1x_3+x_0^2x_2x_3 \\ x_0+x_1 & x_3 & x_0x_1^2x_2+x_1^3x_2+x_0^2x_2x_3+x_1^2x_3^2 \end{pmatrix}$	Z has degree 27 and is the union of a smooth irreducible curve $\mathcal{C}$ of degree 17 and genus 15 with one lines l_1 of degree 7 and another line l_2 of degree 3. The curve $\mathcal{C}$ is 13-secant to l_1 and 11-secant to l_2.
(114(b))	$\begin{pmatrix} x_0 & x_0+x_2+x_3 & x_0^4+x_0^3x_1+x_0^3x_2 \\ x_1 & x_1+x_2+x_3 & x_0^3x_3+x_0x_1^2x_3 \\ x_0+x_1 & x_0+x_2 & x_0^4+x_0^2x_1^2+x_1^3x_2 \\ 2x_0+x_1 & x_1+x_3 & x_1^4+x_0^3x_2+x_1^3x_3 \end{pmatrix}$	Z has degree 27 and is the union of a smooth irreducible curve $\mathcal{C}$ of degree 14 and genus 11 and one line l. The curve $\mathcal{C}$ is 11-secant to l.
(123)	$\begin{pmatrix} x_0+2x_1 & x_1x_2+x_0x_3 & x_0^3+x_0^2x_1+x_0x_1x_3 \\ x_1 & x_1^2+x_1x_3 & x_1^3+x_0^2x_2 \\ x_0+x_1 & x_0x_2+x_1x_3 & x_0x_1x_2+x_0^2x_3 \\ 2x_0+x_1 & x_0^2+x_0x_1 & x_1^2x_2+x_1^2x_3 \end{pmatrix}$	Z has degree 25 and is the union of a smooth irreducible curve $\mathcal{C}$ of degree 14 and genus 11 and one line l. The curve $\mathcal{C}$ is 11-secant to l_1.
(222)	$\begin{pmatrix} x_0x_2+x_0x_3 & x_0x_2 & x_0x_1+x_1x_2+x_0x_3 \\ x_0x_1+x_1x_2 & x_0x_1+x_0x_3 & x_0x_2+x_1x_2+x_0x_3 \\ x_0x_1+x_0x_3 & x_1x_2+x_0x_3 & x_0x_1+2x_0x_2+x_0x_3 \\ x_0x_2+2x_1x_2 & x_0x_1+x_1x_2 & x_0x_2+x_1x_2 \end{pmatrix}$	Z has degree 24 and is the union of an irreducible curve $\mathcal{C}$ of degree 12 and genus 11 and two secant lines l_1 and l_2 of degree 6 each. The curve $\mathcal{C}$ is singular at the intersection point of l_1 and l_2 and is 6-secant to l_1 and 6-secant to l_2.

Remark 4.4.1. Let us emphasize also that it would be interesting to know if, given a determinantal birational map or even a birational map with locally free sheaf of relations, its inverse is also determinantal or has locally free sheaf of relations. Our observations are that the inverse of determinantal birational map of certain type

(for example a quinto-quintic of type (b)) are determinantal of the same type. This leads us to the following conjecture.

Conjecture 4.4.2. *The inverse of determinantal birational map of a certain type is determinantal of the same type.*

4.4.2 Incursion in higher dimension

As a single example in $\mathbb{P}^4$, we consider the determinantal map Φ of the 4×4 minors of the matrix

$$\begin{pmatrix} x_0 & x_1 & x_0 & x_0^2+x_1x_2 \\ x_1 & x_2 & x_1 & x_0x_1+x_1x_3 \\ x_0+x_1 & x_3 & x_2 & x_0x_2+x_1x_4 \\ x_0+2x_1 & x_4 & x_3 & x_0^2+x_0x_3 \\ x_0 & x_0 & x_4 & x_1^2+x_0x_4 \end{pmatrix}.$$

Using Macaulay2, we can say that Φ has multidegree $(1,5,8,5,1)$ (and naive multidegree $(2,7,9,5,1)$). However, the computation of the primary decomposition becomes unsustainable and we cannot describe yet what are the surfaces involved. We conjecture that the general determinantal map of this family has a base locus Z which is the union of an irreducible smooth surface S with resolution:

$$\mathcal{O}_{\mathbb{P}^4}(-9) \longrightarrow \begin{matrix} \mathcal{O}_{\mathbb{P}^4}(-6)^4 \\ \oplus \\ \mathcal{O}_{\mathbb{P}^4}(-8)^2 \end{matrix} \longrightarrow \begin{matrix} \mathcal{O}_{\mathbb{P}^4}(-4) \\ \oplus \\ \mathcal{O}_{\mathbb{P}^4}(-5)^4 \\ \oplus \\ \mathcal{O}_{\mathbb{P}^4}(-7) \end{matrix} \longrightarrow \mathcal{O}_{\mathbb{P}^4} \longrightarrow \mathcal{O}_S \longrightarrow 0$$

and a plane P and that S and P intersect along a curve of degree 7 and arithmetic genus 15.

Pushing further this construction where $P_1 = p_1^*P$ and P is $(n-2)$-plane of $\mathbb{P}^n$ with $n \geq 3$, and where we choose a matrix M such that p_1^*M is general in

$$\mathrm{H}^0\big(\mathcal{I}_{\mathbb{P}_1}(1,1) \oplus \mathcal{O}_{\mathbb{P}^n\times\mathbb{P}^n}(1,1)^{n-2} \oplus \mathcal{I}_{P_1}(2,1)\big),$$

we should be able to construct a determinantal birational map with multidegree $(1, T_1^n, T_2^n, \ldots, T_{n-1}^n, 1)$ where $T_k^n = \binom{n}{k} + \binom{n-2}{k-1}$ are the coefficients of the 3-Pascal triangle (see https://oeis.org/A028262). That is we should be able to construct the following multidegree

Projective space	Multidegree of Φ
$\mathbb{P}^3$	$(1,4,4,1)$
$\mathbb{P}^4$	$(1,5,8,5,1)$
$\mathbb{P}^5$	$(1,6,13,13,6,1)$
$\mathbb{P}^6$	$(1,7,19,26,19,7,1)$

This conjecture is purely experimental and verified from $\mathbb{P}^4$ to $\mathbb{P}^6$.

4.5 Homaloidal complete intersections

To finish this section let us present the following generalisation of homaloidal hypersurfaces.

Definition 4.5.1. Let $n \geq 1$, $1 \leq k \leq n$ and let $g_1, \ldots, g_k \in \mathrm{k}[x_0, \ldots, x_n]$ be k homogeneous polynomials of degree $d_1, \ldots, d_k$. Let denote

$$M = \begin{pmatrix} (g_1)_0 & & \ldots & (g_1)_n \\ \ldots & \ldots & \ldots & \\ (g_k)_0 & & \ldots & (g_k)_n \end{pmatrix}$$

the jacobian matrix associated to $g_0, \ldots, g_k$ that is $(g_j)_i = \frac{\partial g_j}{\partial x_i}$ and let $\phi_0, \ldots, \phi_{\binom{n+1}{k}}$ be the $k \times k$-minors of M. We call the map

$$\begin{aligned} \Phi_{ci} : \quad & \mathbb{P}^n \dashrightarrow \mathbb{P}^{\binom{n+1}{k}-1} \\ & x \dashrightarrow \big(f_0(x) : \ldots : f_N(x)\big) \end{aligned}$$

where $N = \binom{n+1}{k} - 1$, the *polar map of the complete intersection* $(g_1, \ldots, g_k)$. We call the complete intersection homaloidal if Φ_{ci} is birational onto its image.

Remark 4.5.2. When the complete intersection is generated by n homogeneous polynomials $g_1, \ldots, g_n$, we recover a map $\Phi_{ci} : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. This is the reason why this notion of homaloidal complete intersection fits in this chapter about locally free sheaf of relations. Indeed, since the jacobian matrix M of $(g_1 \ \ldots \ g_n)$ has size $n \times (n+1)$, Hilbert-Burch theorem [Eis95, 20.15.b] states that that M is a presentation matrix for its ideal $\mathcal{I}$ of maximal minors if $\mathrm{depth}(\mathcal{I}) \geq 2$. But this is just taking care of the fact that $\mathbb{V}(\mathcal{I})$ has codimension greater than 2. In other words, if $\mathbb{V}(\mathcal{I})$ has expected codimension, the sheaf of relations of $\mathcal{I}$ is split and a locally free resolution of $\mathcal{I}$ reads:

$$0 \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^n}(-a_i) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^n}^{n+1} \longrightarrow \mathcal{I}(\delta) \longrightarrow 0$$

One first result in this context is that, when $n = 2$, there there is no limit in the degree of a homaloidal complete intersection.

Proposition 4.5.3. *Let $r \geq 2$, the complete intersection defined by $x_0x_1, x_1^r + x_0^{r-1}x_2$ of $\mathbb{P}^2$ are homaloidal.*

Proof. The polar map Φ_{ci} is defined by the polynomials

$$\begin{cases} -rx_1^r + (r-1)x_0^{r-1}x_2 \\ -x_0^{r-1}x_1 \\ -x_0^r. \end{cases}$$

It is a computation to show that Φ_{ci} has an inverse defined by

$$-(r-1)x_2^r, -(r-1)x_1x_2^{r-1}, -rx_1^r + x_0x_2^{r-1}.$$

□

Another result is as follows. It is classical result (see [Dés12, Subsection 4.5.2]) that there are three types of quadratic Cremona maps up to left right conjugacy, namely the standard Cremona map with representative $\tau = (x_1x_2, x_0x_2, x_0x_1)$, the quadratic linear system with two indeterminacy points and one base point infinitely near one of the indeterminacy points (see [Dés12, Section 1.2] for the definitions *indeterminacy* and *infinitely near base point*) with representative

$$\tau' = (-2x_1^2 + x_0x_2 + 2x_1x_2, -x_0x_1, -x_0^2 - 2x_0x_1),$$

the quadratic linear system with three base point infinitely near to each others with representative

$$\tau'' = (x_0x_2 - x_1^2, -x_1x_0, x_0^2).$$

We see from the classification of complex homaloidal curves that only τ and τ'' can be realised as the polar of homaloidal curve. Concerning complete intersection of 2 curves in $\mathbb{P}^2$, we have:

Proposition 4.5.4. *All Cremona maps of $\mathbb{P}^2$ of algebraic degree 2 can be realised as the polar map of the complete intersection of two quadrics.*

Proof. The quadratic map τ is the polar of the complete intersection of $q_1 = \frac{x_0^2+x_1^2}{2}$ and $q_2 = \frac{-x_0^2+x_2^2}{2}$.

The quadratic map τ' is the polar of the complete intersection of $q_1' = x_1^2+x_0x_2$ and $q_2' = x_1(x_1 + x_0)$.

The quadratic map τ'' is the polar of the complete intersection of $q_1'' = x_1^2+x_0x_2$ and $q_2'' = x_1x_0$. □

Remark 4.5.5. In general, the polar map Φ_{ci} of k hypersurfaces in $\mathbb{P}^n$ factors through the Grassmanian $\mathrm{Grass}_k(\mathrm{k}^{n+1})$ of k-plane in k^{n+1}. This leads us to the following questions concerning this problem of homaloidal complete intersection: what is the multidegree of Φ_{ci} with respect to the generators of the cohomology of the Grassmanian?

Another perspective would also be to classify homaloidal complete intersections with "small" singularities.

Chapter 5

Free and nearly free sheaves of relations

Let X be the projective plane $\mathbb{P}^2$ over an algebraically closed field k and let $\mathcal{I}$ be an ideal sheaf generated by three global sections ϕ_0, ϕ_1, ϕ_2 of $\mathcal{O}_{\mathbb{P}^2}(\delta)$ for $\delta \geq 1$. In this situation we identify ϕ_0, ϕ_1, ϕ_2 with homogeneous polynomials in the variables x_0, x_1, x_2 of degree δ. Let

$$\begin{tikzcd} \bigoplus_{i\geq 1} \mathcal{O}_{\mathbb{P}^2}(-i)^{a_i} \arrow[rr, "M"] \arrow[dr] & & \mathcal{O}^3_{\mathbb{P}^2} \arrow[r, "(\phi_0\ \phi_1\ \phi_2)"] & \mathcal{I}(\delta) \arrow[r] & 0 \\ & \mathcal{E} \arrow[ur] & & & \end{tikzcd}$$

be the sheafification of a minimal presentation of the ideal $I = (\phi_0, \phi_1, \phi_2)$ and where $\mathcal{E}$ is the image sheaf of M, i.e. the sheaf of relation of $\mathcal{I}$, see Definition 4.0.1.

Since $\mathcal{E}$ is reflexive of rank 2, we have the following result.

Proposition 5.0.1. *[Har80] The sheaf $\mathcal{E}$ is locally free of rank 2.*

Hence, in the perspective of studying the projectivization $\mathbb{X}$ of $\mathcal{I}$, this is a case of interest because we can attempt to relate geometric properties of $\mathcal{E}$, for instance its Chern classes, to geometric properties of $\mathbb{X}$ exactly in the same spirit as in Chapter 4.

Let us explain also our initial motivation for studying this problem. Let $f \in \mathrm{k}[x_0, x_1, x_2]$ be a square free polynomial of degree d and let $\mathcal{I}$ be the jacobian ideal sheaf of f, i.e. the ideal sheaf generated by the partial derivatives $f_i = \frac{\partial f}{\partial x_i}$ of f.

Definition 5.0.2. [Dim15, Definition 2.1] Letting $\mathcal{E}$ be the sheaf of relations of $\mathcal{I}$, the curve $F = \mathbb{V}(f)$ is *free* if $\mathcal{E}$ is split. If $\mathcal{E}$ is split and $\mathcal{E} \simeq \mathcal{O}_{\mathbb{P}^2}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_2)$ then F is *free of exponent* (d_1, d_2).

The curve $F = \mathbb{V}(f)$ is *nearly free* of exponents (d_1, d_2) with $1 \leq d_1 \leq d_2$ if the jacobian ideal $\mathcal{I}$ of F has the following resolution

$$0 \to \mathcal{O}_{\mathbb{P}^2}(-d_2 - 1) \to \mathcal{O}_{\mathbb{P}^2}(-d_1) \oplus \mathcal{O}_{\mathbb{P}^2}(-d_2)^2 \to \mathcal{O}^3_{\mathbb{P}^2} \to \mathcal{I}(d-1) \to 0$$

where d is the degree of F.

See Definition 5.1.1 for the generalisation of these definition. Recall also the definition of Tjurina numbers.

Definition 5.0.3. Let $Z = \mathbb{V}(\mathcal{I})$ be the singular locus of a hypersurface $F = \mathbb{V}(f)$ and let $z \in Z$. Via a change of coordinates, suppose that $z = (1:0:0)$.

The *local Tjurina number* at z, denoted by $\tau_f(Z,z)$ is defined by

$$\tau_f(Z,z) = \mathrm{length}\big(\mathcal{O}_{\mathrm{k}^2,z}/(f_\flat,(f_\flat)_1,(f_\flat)_2)\big) \quad \text{where } (f_\flat)_i = \frac{\partial f_\flat}{\partial x_i}.$$

The *global Tjurina number* of F, denoted by $\tau_f(Z)$ is the sum $\sum \tau_f(Z,z)$ over all $z \in Z$.

A result of A.A. du Plessis and C.T.C.Wall in [dPW99] identifies in particular complex curves F of a given degree d with maximal possible global Tjurina number which are not cones (or else $\mathcal{E}$ has a non trivial factor). These are the free curves of exponents $(1, d-2)$.

But the data of the global Tjurina number of F is equivalent to the data of the second Chern class of $\mathcal{E}$ via the relation

$$c_2(\mathcal{E}) = (d-1)^2 - \tau_f(Z)$$

and let us explain why. Let

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}^3_{\mathbb{P}^2} \xrightarrow{(f_0\ f_1\ f_2)} \mathcal{I}(d-1) \longrightarrow 0 \tag{P6}$$

be the defining exact sequence of $\mathcal{E}$ where $f_i = \frac{\partial f}{\partial x_i}$ for $i \in \{0,1,2\}$. Now, consider a general morphism $\mathcal{O}^2_{\mathbb{P}^2} \to \mathcal{I}(d-1)$, let W be the support of its cokernel. This situation is summed up by the following commutative diagram:

$$\begin{tikzcd}
 & & \mathcal{O}^2_{\mathbb{P}^2} \arrow[r, equal] \arrow[d] & \mathcal{O}^2_{\mathbb{P}^2} \arrow[d] & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[dr] & \mathcal{O}^3_{\mathbb{P}^2} \arrow[r, "(\phi_0\ \phi_1\ \phi_2)"] \arrow[d] & \mathcal{I}(d-1) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_{\mathbb{P}^2} \arrow[r] & \mathcal{O}_W \arrow[r] & 0
\end{tikzcd}$$

Since the global Tjurina number is just the length of the scheme $\mathbb{V}(\mathcal{I})$, we have that $\mathrm{length}(W) = (d-1)^2 - \tau_f(Z)$. Moreover there is an exact sequence $\mathcal{E} \to \mathcal{O}_{\mathbb{P}^2} \to \mathcal{O}_W$ and recall Proposition 1.2.6 that if W has codimension 2 then the class $[W]$ of W in the Chow group $\mathrm{CH}^2(\mathbb{P}^2)$ is the second Chern class of the dual $\mathcal{E}^\vee$ of $\mathcal{E}$. But since $\mathcal{E}$ has rank two, $c_2(\mathcal{E}) = c_2(\mathcal{E}^\vee)$ and, by identifying Chern classes and integers since $\mathrm{CH}^2(\mathbb{P}^2) \simeq \mathbb{Z}$, where we fix a generator to be the class of a point in $\mathbb{P}^2$, we have the relation $c_2(\mathcal{E}) = (d-1)^2 - \tau_f$. Hence the identification of curves with the highest global Tjurina number is equivalent to that of the curves such that $c_2(\mathcal{E})$ is the smallest possible.

Recall that we defined also a generalised Tjurina number in Definition 2.2.21 for any ideal sheaf $\mathcal{I}$ of $\mathbb{P}^2$ generated by three global sections of $\mathcal{O}_{\mathbb{P}^2}(\delta)$. This is just the length of the scheme $\mathbb{V}(\mathcal{I})$. So for us, a main motivation is to elaborate a similar criterion to split the sheaf of relations $\mathcal{E}$. Roughly, one result in this chapter is as follows:

Theorem 5.0.4. *Let $\mathcal{I}$ be an ideal sheaf over $\mathbb{P}^2$ generated by three linearly independant global sections ϕ_0, ϕ_1, ϕ_2 of $\mathcal{O}_{\mathbb{P}^2}(\delta)$ and let $\mathcal{E}$ be the sheaf of relations of $\mathcal{I}$ defined as the kernel of the map $\mathcal{O}^3_{\mathbb{P}^2} \to \mathcal{I}(\delta)$.*

(1) *Then $-c_1(\mathcal{E}) \leq c_2(\mathcal{E})+1$ and equality holds if and only if $\mathcal{E}$ is free of exponents $\big(1, c_2(\mathcal{E})\big)$.*

(2) *In the case $c_1(\mathcal{E}) \leq -5$, $\mathcal{E}$ is nearly free of exponents $\big(1, c_2(\mathcal{E})\big)$ if and only if $-c_1(\mathcal{E}) = c_2(\mathcal{E})$.*

We emphasize that Theorem 5.0.4 is precisely a generalisation of the former result in [dPW99] since we identify free sheaves of exponents $\big(1, c_2(\mathcal{E})\big)$ with sheaves of relations with the smallest second Chern class possible.

The second part of this chapter is the classification of the reduced complex plane curves with respect to the second Chern class of their sheaves of relations, that is, we classify curves of degree d and $c_2(\mathcal{E}) = (d-1)^2 - \tau_f$. To establish it, recall that the polar degree of those curves, i.e. the topological degree of their polar map is equal to $(d-1)^2 - \mu_f$. Hence we simply consider the existing classification of plane curves with given polar degree in [FM12] and we adapt it in the cases where Tjurina numbers differ from Milnor numbers.

5.1 Identification of free and nearly free sheaves

For this subsection, $\mathcal{O}$ stands for $\mathcal{O}_{\mathbb{P}^2}$. For $i \in \{1, 2\}$, we denote by $c_i(\mathcal{E})$ the first and second Chern classes of $\mathcal{E}$ and we identify the Chern classes with their degree in $\mathbb{Z}$.

Definition 5.1.1. A vector bundle $\mathcal{F}$ of rank 2 over $\mathbb{P}^2$ is said to be *free* of exponents (d_1, d_2) if there exists $(d_1, d_2) \in \mathbb{N}^{*2}$ such that $\mathcal{F} \simeq \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2)$.

It is said to be *nearly free* of exponents (d_1, d_2) if it has a graded free resolution of the form:

$$0 \longrightarrow \mathcal{O}(-d_2-1) \longrightarrow \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2)^2 \xrightarrow{(\phi_0\ \phi_1\ \phi_2)} \mathcal{F} \longrightarrow 0.$$

Definition 5.1.2 ([DS15])**.** In the case where $\phi_0 = f_0$, $\phi_1 = f_1$, $\phi_2 = f_2$ are the partial derivatives of a given squarefree polynomial $f \in \mathrm{k}[x_0, x_1, x_2]$, the curve $F = \{f = 0\}$ is called free (resp. nearly-free) if $\mathcal{E}$ in (P6) is free (resp. nearly-free).

As we explained in the introduction of this section, the maximality of the Tjurina number is equivalent to the minimality of the second Chern class $c_2(\mathcal{E})$ of the vector bundle $\mathcal{E}$ associated to F. Theorem 5.0.4 is thus a generalisation of the result of du Plessis-Wall. We emphasize that in this case $c_1(\mathcal{E})$ is negative and $c_2(\mathcal{E})$ is positive.

Proof of Theorem 5.0.4. We denote by c_1 and c_2 respectively the first Chern class $c_1(\mathcal{E})$ and the second Chern class $c_2(\mathcal{E})$ of $\mathcal{E}$. We let $c = -1 - c_1 \geq 0$ and

$$m = \min\{t \in \mathbb{Z},\ \mathrm{H}^0\big(\mathbb{P}^2, (\mathcal{E}(t)\big) \neq 0\}.$$

(1) Assume that $c_2 \leq c$. We are going to show that the only possibility is that $c_2 = c$ and $m = 1$. First, $m > 0$ since otherwise, if $0 \neq s \in \mathrm{H}^0(\mathbb{P}^2, \mathcal{E})$ we would have had $\mathcal{E} \simeq \mathcal{O} \oplus \mathcal{O}(-1-c)$ which contradicts the fact that $c_2 > 0$.

Now, let $s \in \mathrm{H}^0\left(\mathbb{P}^2, \mathcal{E}(m)\right)$ be a non zero section. Since m is minimal, we have the following exact sequence:

$$0 \longrightarrow \mathcal{O}(-m) \longrightarrow \mathcal{E} \longrightarrow \mathcal{I}_L(m-1-c) \longrightarrow 0 \tag{E7}$$

where $L \subset \mathbb{P}^2$ is a 0-dimensional subscheme of length $l \geq 0$. It is a computation to show that $l = c_2 - m(c+1-m) \geq 0$, and since $c_2 \leq c$, we have

$$c(1-m) \geq m(1-m). \tag{5.1.1}$$

So

(i) if $m = 1$, then $l = 0$, i.e. $\mathcal{I}_L(m-1-c) = \mathcal{O}(m-1-c)$ and the sequence (E7) splits showing that $\mathcal{E} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-c)$,

(ii) if $m \geq 2$ then $m \geq c$.

Now, assume by contradiction that $m \geq 2$. First, it follows from the Riemann-Roch formula that:

$$\chi\big(\mathcal{E}(1)\big) = \frac{8 - 2c_2 - 3c + c^2}{2} \geq \frac{8 - 5c + c^2}{2}.$$

Hence $\chi\big(\mathcal{E}(1)\big) > 0$ for all c. On the other hand, since $m \geq 2$, by (ii), $m \geq c$ and we have $\mathrm{H}^0\big(\mathbb{P}^2, \mathcal{E}(1)\big) = \mathrm{H}^2\big(\mathbb{P}^2, \mathcal{E}(1)\big) = 0$ where the second vanishing follows from the first, using Serre-duality $\mathrm{H}^2\big(\mathbb{P}^2, \mathcal{E}(1)\big) \simeq \mathrm{H}^0\big(\mathbb{P}^2, \mathcal{E}(c-3)\big)^\vee$. These two vanishings contradict the fact that $\chi\big(\mathcal{E}(1)\big) > 0$. Summing up, if $c_2 \leq c$, the only possibility is $c_2 = c$ and then $\mathcal{E} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-c)$ which completes the proof of (1).

(2) It is a computation to show that if $\mathcal{E}$ is nearly free of exponents $(1, c_2)$, then $c_2 = c + 1 = -c_1$. Now, we assume that $c_2 = c + 1$ and that $c \geq 4$ and we show that $\mathcal{E}$ is nearly-free of exponents $(1, c_2)$. In this case, from (E7) we have the inequality

$$c(1-m) \geq m(1-m) - 1 \tag{5.1.2}$$

so:

(i) $m \geq 3$ implies $m \geq c$ (because in this case (5.1.2) is equivalent to $c \leq m - \frac{1}{1-m}$) and thus $\mathrm{H}^2\big(\mathbb{P}^2, \mathcal{E}(1)\big) = \mathrm{H}^0\big(\mathbb{P}^2, \mathcal{E}(1)\big) = 0$. But the Riemann-Roch formula implies that

$$\chi\big(\mathcal{E}(1)\big) = \tfrac{(c-2)(c-3)}{2},$$

hence $\chi\big(\mathcal{E}(1)\big) > 0$ for $c \geq 4$. As above this leads to a contradiction and so this case does not occur.

(ii) $m = 2$ implies $c \leq 3$, a case excluded by the assumption $c \geq 4$.

(iii) $m = 1$ implies that $l = 1$ where l is the length of the scheme L as in the exact sequence (E7). Now, using the resolution of a point p in $\mathbb{P}^2$, we get the following diagram:

$$\begin{tikzcd}
 & & & 0 & \\
0 \arrow[r] & \mathcal{O}(-1) \arrow[r, "\alpha"] & \mathcal{E} \arrow[r] & \mathcal{I}_p(-c) \arrow[u] \arrow[r] & 0 \\
 & & & \mathcal{O}(-1-c)^2 \arrow[u] \arrow[ul, dashed, "\beta"] & \\
 & & & \mathcal{O}(-2-c) \arrow[u] & \\
 & & & 0 \arrow[u] &
\end{tikzcd}$$

where the existence of β is provided by the vanishing of $\mathcal{E}\mathrm{xt}^1(\mathcal{O}(-1-c)^2, \mathcal{O}(-1))$ (see also [MV17] for more details in this direction). Since $\mathcal{E}$ is locally free of rank 2, the complex (E8) provides a locally free resolution of $\mathcal{E}$ showing that $\mathcal{E}$ is nearly-free of exponent $(1, -1-c)$, that is, $\mathcal{E}$ has the resolution:

$$0 \longrightarrow \mathcal{O}(-c-2) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-c-1)^2 \longrightarrow \mathcal{E} \longrightarrow 0. \qquad \text{(E8)}$$

□

As an application we recover [DHS12, Corollary 2.6] but with a different proof. Recall that $\mathcal{I}$ is said to be of *linear type* if the projectivization $\mathbb{X}$ of $\mathcal{I}$ coincide with the **Proj** $\tilde{X}$ of the Rees algebra of $\mathcal{I}$ (see Definition 2.2.10). With the definitions of Milnor and Tjurina numbers given in Definition 2.2.21, recall that the ideal sheaf $\mathcal{I}_Z$ of a 0-dimensional scheme Z in a quasi-projective variety X is of linear type if and only if $\mu(Z) = \tau(Z)$.

Corollary 5.1.3. *Let $\mathcal{I} = (\phi_0, \phi_1, \phi_2)$ be an ideal sheaf generated by three homogeneous polynomials of degree δ without common factor. Assume that $\mathcal{I}$ is of linear type then the associated map Φ is birational only if $\delta \leq 2$.*

Proof. Indeed, letting $\mathcal{E}$ be as in (P6), we have that $c_2(\mathcal{E}) = d_t(\Phi)$. But $c_1 = -\delta$ so the only possibility to have $d_t(\Phi) = 1$ is that $\delta \leq 2$. □

Let focus on another special resolution of a vector bundle $\mathcal{E}$ of rank 2 over $\mathbb{P}^2$. In the following, we assume that $\mathcal{E} \subset \mathcal{O}_{\mathbb{P}^2}^r$ for a given $r \in \mathbb{N}$ in order to have $\mathrm{h}^0(\mathbb{P}^2, \mathcal{E}) = 0$. Recall that such a bundle can be thought as in the exact sequence (P6).

Definition 5.1.4. We say that $\mathcal{E}$ is *almost nearly free* of exponent (d_1, d_2) if it has the following resolution:

$$0 \longrightarrow \mathcal{O}(-d_2-2) \longrightarrow \mathcal{O}(-d_1) \oplus \mathcal{O}(-d_2) \oplus \mathcal{O}(-d_2-1) \longrightarrow \mathcal{E} \longrightarrow 0.$$

Example 5.1.5. We say that a curve in $\mathbb{P}^2$ is *almost nearly free* if $\mathcal{E}$ in the exact sequence (P6) is almost nearly free. For instance, the curve

$$F = \mathbb{V}\big((x_1^4 + x_0^3 x_2)(x_2 + x_1)\big)$$

is almost nearly free of exponents $(2, 3)$ (a result we find by computing a free resolution of the jacobian ideal of F via Macaulay2).

Proposition 5.1.6. *Let $\mathcal{E}$ be a rank 2 vector bundle over $\mathbb{P}^2$ with $c_1(\mathcal{E}) = -1 - c$ and with $1 \leq c_2(\mathcal{E}) = c + 2$.*

If $c \geq 5$ then $\mathcal{E}$ is almost nearly free of exponent $(1, c_2(\mathcal{E}) - 1)$.

We give the proof of such a statement to show how the definition of almost nearly free vector bundles is analogous to the one of nearly free bundles.

Proof. Let $m = \min\{t \in \mathbb{Z}\ ,\ \mathrm{h}^0(\mathbb{P}^2, \mathcal{E}(t)) \neq 0\}$. Using the sequence (E7), we have the inequality: $c(1-m) + 2 \geq m(1-m)$ that is

$$c + \frac{2}{1-m} \leq m$$

From this inegality, we get:

(i) if $m \geq 4$ then $c \leq m$

(ii) if $m = 3$ or $m = 2$ then $c \leq 4$

(iii) if $m = 1$, then $l = 2$ where l is the length of the scheme L in the exact sequence (E7). Since the resolution of such a scheme L in $\mathbb{P}^2$ is:

$$0 \longrightarrow \mathcal{O}(-3) \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{I}_L \longrightarrow 0,$$

as in the proof of Theorem 5.0.4, one has:

$$0 \rightarrow \mathcal{O}(-c-3) \rightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-c-1) \oplus \mathcal{O}(-c-2) \rightarrow \mathcal{E} \rightarrow 0.$$

But now if $c \geq 5$ the case (iii) cannot happen. Indeed, if it would happen, one would have $\mathrm{h}^0(\mathbb{P}^2, \mathcal{E}(1)) = \mathrm{h}^0(\mathbb{P}^2, \mathcal{E}(c-1)) = 0$. The Riemann-Roch formula gives then

$$\chi(\mathcal{E}(1)) = \frac{(c-1)(c-4)}{2} > 0$$

which is impossible. So $\mathcal{E}$ is almost nearly free of exponent $(1, c+1)$. □

Remark 5.1.7. We could want to find almost nearly free curves of degree $d \geq 7$ with exponent $(1, c_2(\mathcal{E}) - 1)$ But as it is stated in [dPW99, Th.3.2] in the case where there is a first syzygy of degree one, that is with the notation of their paper, when $r = 1$, we have the following inequality:

$$(d-1)(d-2) \leq \tau \leq (d-1)(d-2) + 1$$

where τ is the Tjurina number of the curve. In our case, we would want a curve such that $\tau = (d-1)^2 - d = d^2 - 3d + 1 \leq d^2 - 3d - 2 = (d-1)(d-2)$ which is impossible.

5.2 Classification of curves by Tjurina numbers

In this section, we assume that the base field k has characteristic 0. In this case, we propose a classification based on [FM12, Theorem 3.3] and [FM12, Theorem 3.4] of plane curves which are not cones with respect to the second Chern class of the sheaf of relations $\mathcal{E}$ of the jacobian ideal sheaf of the curve. This classification is the same except when the jacobian ideal is not of linear type, i.e. when Tjurina and Milnor numbers of the curve differ. In the table, we specify by (d_1, d_2)-f. or (d_1, d_2)-n.f. if the curves are respectively free of exponent (d_1, d_2) or nearly free of exponent (d_1, d_2).

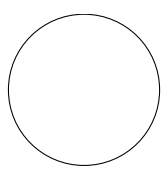
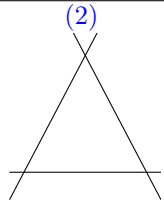
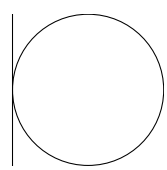
All the computation of Milnor and Tjurina numbers and of resolutions of the jacobian ideals of the curves (in order to establish in which case the curves are free or nearly free) were made by a case by case analysis of the explicit equations of the curves using Macaulay2.

Theorem 5.1. *A reduced plane curve with $c_2(\mathcal{E})$ equal to 1 is one of the following:*

(1) *a smooth conic,*

(2) *three lines in general position*

(3) *the union of a smooth conic with one of its tangent*

(1)	(2)	(3)
		
$\mathbb{V}(x_1^2 + x_0x_2)$	$\mathbb{V}(x_0x_1x_2)$	$\mathbb{V}(x_0(x_1^2 + x_0x_2)$
$(1,1)$-*n.f.*	$(1,1)$-*f.*	$(1,1)$-*f.*

Proof. Let C be a curve with $c_2(\mathcal{E}) = 1$ and let denote by $\mathcal{E}$ its sheaf of relation as in (P6). By Theorem 5.0.4, the algebraic degree d of C is $0, 1, 2$ or 3. Then, a case-by-case study gives the classification. □

We obtain the classification of the curves with $c_2(\mathcal{E}) = 2$ as follows. In [FM12, Theorem 3.3] the two authors gave the complete classification of curves with polar degree 2. Hence, the curves with jacobian ideal of linear type in this classification are curves with $c_2(\mathcal{E}) = 2$. Since we focus on dominant polar maps, it only remains to add the eventual curves of degrees between 0 to 4 with jacobian ideal not of linear type with polar degree 1 (there is none).

Theorem 5.2. *A reduced plane curve with $c_2(\mathcal{E}) = 2$ with dominant polar map is one of the following:*

(1) *three concurrent lines and a fourth line not meeting the center point,*

(2) *a smooth conic and a secant line,*

(3) *a smooth conic, a tangent and a line passing through the tangency point,*

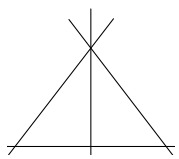
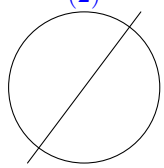
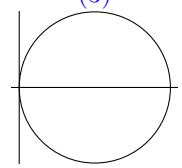
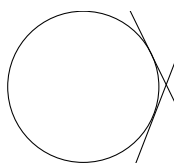
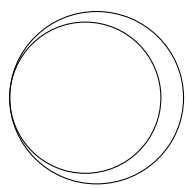
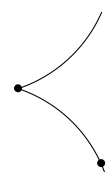
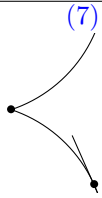
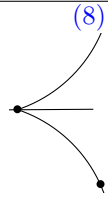
(4) *a smooth conic and two tangent lines,*

(5) *two smooth conics meeting at a single point,*

(6) *an irreducible cuspidal cubic,*

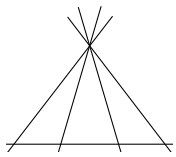
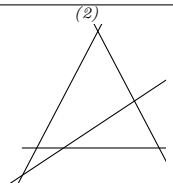
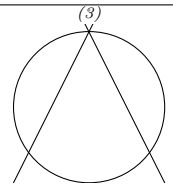
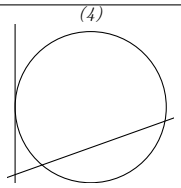
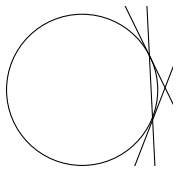
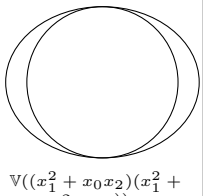
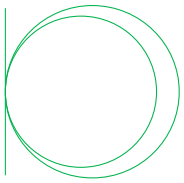
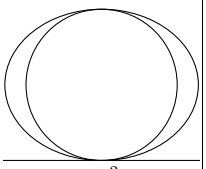
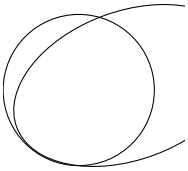
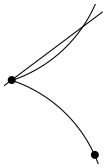
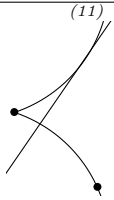
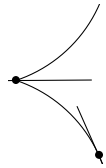
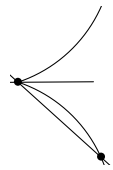
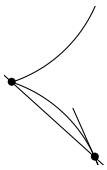
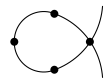
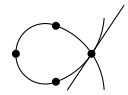
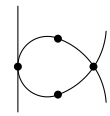
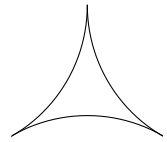
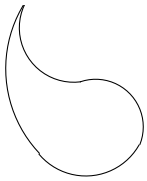
(7) *an irreducible cuspidal cubic and its tangent at the smooth flex point*

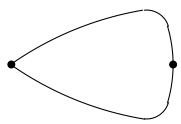
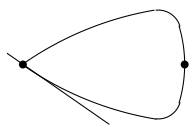
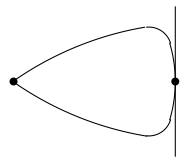
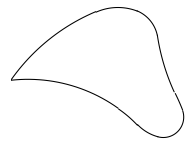
(8) *an irreducible cuspidal cubic and its tangent at the cusp,*

(1)  $\mathbb{V}(x_0x_1x_2(x_0+x_1))$ $(1,2)$-*f.*	(2)  $\mathbb{V}(x_0(x_0+x_1)(x_1^2+x_0x_2))$ $(1,2)$-*f.*	(3)  $\mathbb{V}(x_0(x_1^2+x_0x_2)$ $(1,1)$-*f.*
(4)  $\mathbb{V}(x_0x_2(x_1^2+x_0x_2))$ $(1,2)$-*f.*	(5)  $\mathbb{V}((x_1^2+x_0x_2)(x_1^2+x_0x_2+x_0^2))$ $(1,2)$-*f.*	(6)  $\mathbb{V}(x_1^3+x_0^2x_2)$ $(1,2)$-*n.f.*
(7)  $\mathbb{V}(x_2(x_1^3+x_0^2x_2))$ $(1,2)$-*f.*	(8)  $\mathbb{V}(x_0(x_1^3+x_0^2x_2))$ $(1,2)$-*f.*	

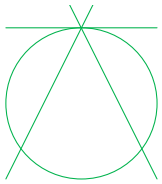
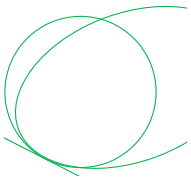
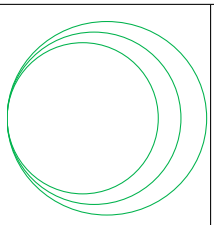
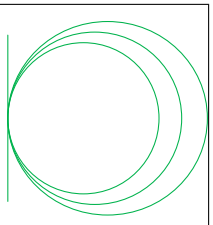
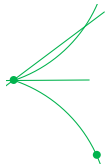
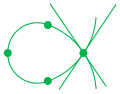
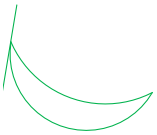
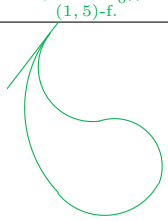
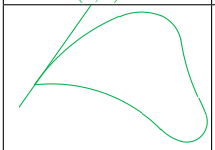
As in [FM12, Theorem 3.4], we also provide the classification of plane curves such that $c_2(\mathcal{E}) = 3$ and with a dominant polar map. We picture in green the plane curve whose jacobian ideal is not of linear type.

Theorem 5.3. *A reduced plane curve with $c_2(\mathcal{E}) = 3$ is one of the types in the following tabular (up to a change of coordinates).*

(1)  $\mathbb{V}(x_0x_1x_2(x_0+x_1)(x_0+2x_1))$ $(1,3)$-*f.*	*(2)*  $\mathbb{V}(x_0x_1x_2(x_0+x_1+x_2))$ $(2,2)$-*n.f.*	*(3)*  $\mathbb{V}((x_1^2+x_0x_2)(x_0+x_1)(x_0+2x_1))$ $(2,2)$-*n.f.*	*(4)*  $\mathbb{V}((x_1^2+x_0x_2)x_0(x_0+x_2))$ $(2,2)$-*n.f.*
(5)  $\mathbb{V}(x_0x_2x_1(x_1^2+x_0x_2))$ $(1,3)$-*f.*	*(6)*  $\mathbb{V}((x_1^2+x_0x_2)(x_1^2+2x_0x_2))$ $(1,3)$-*n.f.*	*(7)*  $\mathbb{V}(x_0(x_1^2+x_0x_2)(x_1^2+x_0x_2+x_0^2))$ $(1,3)$-*f.*, $d_t=2$	*(8)*  $\mathbb{V}(x_0(x_1^2+x_0x_2)(x_1^2+2x_0x_2))$ $(1,3)$-*f.*
(9)  $\mathbb{V}((x_1^2+x_0x_2)(x_1^2+x_0x_2+x_1x_2))$ $(2,2)$-*n.f.*	*(10)*  $\mathbb{V}((x_1+x_0)(x_1^3+x_0^2x_2))$ $(1,3)$-*n.f.*	*(11)*  $\mathbb{V}((x_1^3+x_0^2x_2)(x_2+3x_1-2x_0))$ $(2,2)$-*n.f.*	*(12)*  $\mathbb{V}(x_0x_2(x_1^3+x_0^2x_2))$ $(1,3)$-*f.*
(13)  $\mathbb{V}(x_0x_1(x_1^3+x_0^2x_2))$ $(1,3)$-*f.*	*(14)*  $\mathbb{V}(x_1x_2(x_1^3+x_0^2x_2))$ $(1,3)$-*f.*	*(15)*  $\mathbb{V}(x_1^2x_2-x_0^2(x_0+x_2))$	*(16)*  $\mathbb{V}((x_0-x_1)(x_1^2x_2-x_0^2(x_0+x_2)))$ $(2,2)$-*n.f.*
(17)  $\mathbb{V}(x_2(x_1^2x_2-x_0^2(x_0+x_2)))$ $(2,2)$-*n.f.*	*(18)*  $\mathbb{V}(8x_0x_1^3-3x_1^4+12x_0x_1^2x_2-4x_1^3x_2+4x_0^2x_2^2)$ $(2,2)$-*n.f.*	*(19)*  $\mathbb{V}(x_1^4-2x_0x_1^2x_2-x_1^3x_2+x_0^2x_2^2)$ *singular points :* $(1:0:0)$, $(0:0:1)$ $(2,2)$-*n.f.*	*(20)*  $\mathbb{V}(x_1^4-2x_0x_1^2x_2-x_1x_2^3+x_0^2x_2^2)$ *singular point :* $(1:0:0)$, $(2,2)$-*n.f.*

(21)	*(22)*	*(23)*	*(24)*
			
$\mathbb{V}(x_1^4 - x_0x_2^3)$ *singular point :* $(1:0:0)$, $(1,3)$-*n.f.*	$\mathbb{V}(x_2(x_1^4 - x_0x_2^3)$ $(1,3)$-*f.*	$\mathbb{V}(x_0(x_1^4 - x_0x_2^3))$ $(1,3)$-*f.*	$\mathbb{V}(x_1^4 - x_1^3x_2 + x_0x_2^3)$ *singular point :* $(1:0:0)$, $(2,2)$-*n.f.*

We end this chapter by giving the curves with polar degree 3 and naive polar degree $c_2(\mathcal{E}) = 4$.

			
$\mathbb{V}((x_1^2 + x_0x_2)(x_0 + x_1)(x_0 + 2x_1)x_0)$ $(2,2)$-f.	$\mathbb{V}(x_2(x_1^2+x_0x_2)(x_1^2+x_0x_2 + x_1x_2))$ $(2,2)$-f.	$\mathbb{V}((x_1^2 + x_0x_2)(x_1^2 + x_0x_2 + x_0^2)(x_1^2 + x_0x_2 + 2x_0^2))$ $(1,4)$-f.	$\mathbb{V}(x_0(x_1^2+x_0x_2)(x_1^2+x_0x_2 + x_0^2)(x_1^2 + x_0x_2 + 2x_0^2))$ $(1,5)$-f.
			
$\mathbb{V}(x_0(x_0 + x_1)(x_1^3 + x_0^2x_2))$ $(2,2)$-f.	$\mathbb{V}((x_0 - x_1)(x_0 + x_1)(x_1^2x_2 - x_0^2(x_0 + x_2)))$ $(2,2)$-f.	$\mathbb{V}(x_2(x_1^4 - 2x_0x_1^2x_2 - x_1^3x_2 + x_0^2x_2^2))$ $(2,2)$-f.	$\mathbb{V}(x_2(x_1^4 - 2x_0x_1^2x_2 - x_1x_2^3 + x_0^2x_2^2))$ $(2,2)$-f.
			
$\mathbb{V}(x_2(x_1^4 - x_1^3x_2 + x_0x_2^3))$ $(2,2)$-f.			

Chapter 6

Zero-dimensional base locus

In this section, we focus on the naive multidegree of a rational map $\Phi : X \dashrightarrow \mathbb{P}^n$ with a zero-dimensional base locus Z in a smooth quasi-projective variety X of dimension n. Let us first present the problem we deal with. As we explained in Chapter 4 in the case $X = \mathbb{P}^n$, if the sections $\phi_0, \ldots, \phi_n$ defining $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ are the $n+1$ minors of a matrix M of size $(n+1) \times n$, it is particularly easy to compute its naive multidegree. Actually this can be done in two ways. The first one is to consider a presentation of the base ideal sheaf $\mathcal{I}_Z$

$$0 \ \to \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^n}(-a_i) \xrightarrow{M} \mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{I}_Z(\delta) \to \ 0.$$

The nth naive projective degree of Φ is then the length of the intersection of n general generators $\sum_{i=0}^{n} \lambda_i \phi_i$ where $\lambda_i \in \mathrm{k}$ for any $i \in \{0, \ldots, n\}$ after removing the points already in Z (see Section 1.3).

Hence the nth projective degree of Φ is the length of the scheme $\mathbb{V}(s)$ defined in the following exact sequence:

$$\mathcal{O}_{\mathbb{P}^n}^{n} \to \mathcal{I}_Z(\delta) \to \mathcal{O}_{\mathbb{V}(s)} \to \ 0$$

where the morphism $\mathcal{O}_{\mathbb{P}^n}^{n} \to \mathcal{I}_Z(\delta)$ is general. Now consider this morphism in the following commutative diagram:

$$\begin{tikzcd}
 & & 0 \arrow[d] & & \\
 & & \mathcal{O}_{\mathbb{P}^n}^{n} \arrow[r, equal] \arrow[d] & \mathcal{O}_{\mathbb{P}^n}^{n} \arrow[d] & \\
0 \arrow[r] & \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^n}(-a_i) \arrow[r, "M"] \arrow[dr] & \mathcal{O}_{\mathbb{P}^n}^{n+1} \arrow[r] \arrow[d] & \mathcal{I}_Z(\delta) \arrow[r] \arrow[d] & 0 \\
 & & \mathcal{O}_{\mathbb{P}^n} \arrow[r] \arrow[d] & \mathcal{O}_s \arrow[r] \arrow[d] & 0 \\
 & & 0 & 0 &
\end{tikzcd}$$

So $\mathbb{V}(s)$ is also the support of the cokernel of the map $\mathcal{E} = \bigoplus_{i=1}^{n} \mathcal{O}_{\mathbb{P}^n}(-a_i) \to \mathcal{O}_{\mathbb{P}^n}$. In other words, the class $[\mathbb{V}(s)]$ of $\mathbb{V}(s)$ in the Chow ring of $\mathbb{P}^n$ is the nth Chern class of the dual $\mathcal{E}^\vee$ of $\mathcal{E}$, i.e. s is a cosection of $\mathcal{E}$. Since $\mathcal{E}$ is in particular locally free, we have that s is the nth Chern class of $\mathcal{E}$. As such, its length (as a 0-cycle) is particularly easy to compute in this case, it is $\prod_{i=0}^{n} a_i$.

The second way to compute this projective degree is to consider the projectivization $\mathbb{X}$ of $\mathcal{I}_Z$ in $\mathbb{P}^n \times \mathbb{P}^n$. Since it is a complete intersection of cohomological class $\prod_{i=1}^{n} (a_i h_1 + h_2)$ in $\mathrm{CH}(\mathbb{P} \times \mathbb{P}^n)$, we recover that the nth naive projective degree is $\prod_{i=0}^{n} a_i$.

The main goal of this chapter is to establish if those two ways of computing the naive projective degrees, namely considering cosections of $\mathcal{E}$ or the projectivization $\mathbb{X}$, always coincide even if $\mathcal{E}$ is not locally free. We summarize our result as follow, see Theorem 6.1.1 for a more precise statement.

Theorem 6.0.1. *In the case when the base locus Z is 0-dimensional, the length of the zero scheme of a cosection of the kernel $\mathcal{E}$ of the evaluation map $\mathcal{O}_{\mathbb{P}^n}^{n+1} \to \mathcal{I}(\delta)$ is equal to the nth naive projective degree.*

In the case when the base locus Z is 0-dimensional, we emphasize that the possible torsion components of $\mathbb{X}$ only interfere with the nth projective degree, that is, projective degrees and naive projective degrees coincide up to the nth projective degree. That is why we only consider two naive topological degrees, see Definition 6.1.4 and Definition 6.1.5 which have to be understood as the two ways of computing the nth projective degree of Φ.

6.1 First and second naive degrees

We let X be an n-dimensional smooth quasi-projective variety over k and we let $\Phi : X \dashrightarrow \mathbb{P}^n$ be a rational map with zero-dimensional base locus Z determined by a $n+1$-dimensional subspace V of global sections of a line bundle $\mathcal{L}$ over X. Our aim is to read off the topological degree $d_n(\Phi)$ of Φ from properties of the ideal sheaf $\mathcal{I}$ of Z, more precisely from the sheaf of relations $\mathcal{E}$ defined as the kernel of the canonical evaluation map $ev : \mathcal{O}_X \otimes \mathrm{V} \to \mathcal{I} \otimes \mathcal{L}$ (Definition 4.0.1). We set also the notation as in the following diagram

$$\begin{tikzcd}
 & \mathbb{P}^n_X \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\
 & \mathbb{X} \arrow[u, hook, "\iota"] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & \tilde{X} \arrow[u, hook] \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\
X \arrow[rr, dashed, "\Phi"] & & \mathbb{P}^n
\end{tikzcd} \tag{D1}$$

where $\mathbb{X}$ is the projectivization of $\mathcal{I}_Z$ and $\tilde{X}$ is the blow-up of X along Z (recall Proposition 2.1.8 that $\tilde{X}$ is isomorphic to the graph Γ of Φ). By construction the topological degree of Φ is equal to that of the restriction to $\tilde{X}$ of the lift $\pi_2 : \mathbb{X} \to \mathbb{P}(\mathrm{V})$ of Φ. In other word $d_t(\Phi) = \deg\big(c_1(\mathcal{O}_{\mathbb{X}}(1)|_{\tilde{X}})^n\big)$.

In this context, we can also consider two other related notions of "naive" topological degrees: the degree $\deg\big(c_1(\mathcal{O}_{\mathbb{X}}(1))^n\big)$ of π_2, which is the definition of the nth naive projective degree of Φ and the algebraic degree of Φ minus the length of Z. In Proposition 6.2.2 *(i)* we show that the second one coincides with the degree of the 0-cycle $[\mathbb{V}\big({}^{\mathrm{c}}\mathrm{s}(\mathcal{E})\big)]$ associated to the scheme of zeros of a general cosection ${}^{\mathrm{c}}\mathrm{s}(\mathcal{E}) : \mathcal{E} \to \mathcal{O}_X$ of $\mathcal{E}$. Our main result, proven in Subsection 6.1.3, asserts in particular that these two naive topological degrees coincide. It also elucidates the relation between these degrees and the topological degree of Φ:

Theorem 6.1.1. *With the notation above, $\mathbb{X}$ is equidimensional of dimension n and $[\mathbb{V}\big({}^{\mathrm{c}}\mathrm{s}(\mathcal{E})\big)] = \pi_{1*}c_1\big(\mathcal{O}_{\mathbb{X}}(1)\big)^n$. As a consequence*

$$d_n(\Phi) = \deg\big([\mathbb{V}({}^{\mathrm{c}}\mathrm{s}(\mathcal{E}))]\big) - \deg\big(c_1(\mathcal{O}_{\mathbb{X}}(1)|_{\mathbb{T}_Z})^n\big).$$

where $\mathbb{T}_Z$ is the torsion part of $\mathbb{X}$, see Notation 2.2.20.

Recall that the two classical invariants of singularities of a hypersurface $F = \{f = 0\}$ is the global Tjurina number $\tau_f(Z)$ of F and the global Milnor number $\mu_f(Z)$ (see Definition 16 and Definition 8). Actually, both Milnor and Tjurina numbers depend on the scheme structure of the singular locus, and, in this sense, they can be defined also for zero-dimensional subschemes Z unrelated to singular hypersurfaces. Having this in mind, recall the result Theorem 10

Theorem 6.1.2. *Let $f \in \mathbb{C}[x_0, \ldots, x_n]$ be a square free homogeneous polynomial of degree d and let Φ_f be the polar map of $F = \mathbb{V}(f) \subset \mathbb{P}^n$. Assuming that F has finite base locus, we have:*

$$d_n(\Phi_f) = (d-1)^n - \mu_f(Z) \tag{6.1.1}$$

In comparison to (6.1.1), we can formulate the following result.

Corollary 6.1.3. *Formula (6.1.1) holds for any 0-dimensional subscheme Z defined by $n+1$ global sections of a line bundle $\mathcal{L}$ over a smooth quasi-projective n-variety X and over any algebraically closed field.*

This corollary follows from the observation that Tjurina numbers compute the degree of $c_1\big(\mathcal{O}_{\mathbb{X}}(1)\big)^n$ whereas Milnor numbers compute the degree of $c_1\big(\mathcal{O}_{\tilde{X}}(1)\big)^n$. As an immediate application we recover the identity (0.0.1) from the equalities

$$\deg\big([\mathbb{V}({}^{\mathrm{c}}\mathrm{s}(\mathcal{E}))]\big) = (d-1)^n - \tau(Z) \quad \text{and} \quad \deg\big(c_1(\mathcal{O}_{\mathbb{X}}(1)|_{\mathbb{T}_Z})^n\big) = \mu(Z) - \tau(Z)$$

where $\tau(Z)$ and $\mu(Z)$ are the generalised Tjurina and Milnor numbers.

In our setting, recall that $\mathbb{X}$ is equidimensional of dimension n (see Proposition 2.2.19) so $c_1(\mathcal{O}_{\mathbb{P}^n_X}(1)|_{\mathbb{X}})^n$ is a 0-cycle on $\mathbb{X}$ and we can set the following definition.

Definition 6.1.4. With the notation in (D1) the degree of $c_1(\mathcal{O}_{\mathbb{P}^n_X}(1)|_{\mathbb{X}})^n$ is called the *first naive topological degree* of Φ.

Intuitively, the difference between the first naive topological degree and the actual topological degree reflects a difference between the symmetric algebra and the Rees algebra, see Proposition 6.2.2 below for a precise statement.

Now, let $\mathcal{E}$ be the kernel of the evaluation map $ev : \mathcal{O}_X^{n+1} \to \mathcal{I} \otimes \mathcal{L}$ and let $\alpha : \mathcal{O}_X^{n+1} \to \mathcal{O}_X$ be a general map. Since $\mathcal{E}$ has rank n, the zero locus $\mathbb{V}({}^{\mathrm{c}}\mathrm{s}_\alpha)$ of the composition ${}^{\mathrm{c}}\mathrm{s}_\alpha = \alpha \circ \gamma$ is a 0-dimensional subscheme of X.

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{E} \arrow[r, "\gamma"] \arrow[dr, "{}^{\mathrm{c}}\mathrm{s}_\alpha"'] & \mathcal{O}_X^{n+1} \arrow[r, "ev"] \arrow[d, "\alpha"] & \mathcal{I} \otimes \mathcal{L} \arrow[r] & 0 \\ & & \mathcal{O}_X & & \end{tikzcd}$$

In the proof of Theorem 6.1.1, we will establish in particular that the cycle class $[\mathbb{V}({}^{\mathrm{c}}\mathrm{s}_\alpha)]$ of $\mathbb{V}({}^{\mathrm{c}}\mathrm{s}_\alpha)$ is independent on the choice of a general map α so, anticipating, we set the following definition.

Definition 6.1.5. The *second naive topological degree* of Φ is the degree of the 0-cycle $[\mathbb{V}\big({}^{\mathrm{c}}\mathrm{s}(\mathcal{E})\big)]$ of a general cosection ${}^{\mathrm{c}}\mathrm{s}(\mathcal{E})$ of $\mathcal{E}$.

Remark 6.1.6. If $\mathcal{E}$ is locally free, $[\mathbb{V}\big({}^{\mathrm{c}}\mathrm{s}(\mathcal{E})\big)]$ simply coincides with the top Chern class $c_n(\mathcal{E}^\vee)$ of $\mathcal{E}^\vee$. This is no longer true when $\mathcal{E}$ is not locally free. For instance the sheaf $\mathcal{E}$ of relations of the ideal sheaf $\mathcal{I} = (x_1^2 - x_1x_3, x_2^2 - x_2x_3, x_1x_2, x_0x_3)$ of $\mathbb{P}^3$ satisfies $c_3(\mathcal{E}^\vee) = 4$ whereas $\deg\big([\mathbb{V}({}^{\mathrm{c}}\mathrm{s}(\mathcal{E}))]\big) = 2$ as we can check from the resolution of $\mathcal{E}$:

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-3)^2 \longrightarrow \mathcal{O}_{\mathbb{P}^3}(-1)^2 \oplus \mathcal{O}_{\mathbb{P}^3}(-2)^3 \longrightarrow \mathcal{E} \longrightarrow 0.$$

6.1.1 Importance of the subregularity of the symmetric algebra

Recall the settings of Theorem 6.1.1, we assume that $n \geq 2$, $\mathrm{codim}(Z) = n$ and that the map Φ is dominant.

By definition, the first naive topological degree is the length of the 0-scheme W of a general section of $\mathcal{O}_{\mathbb{X}}(1)^n$. Our strategy to show Theorem 6.1.1 is now to push forward the following exact sequence:

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{O}_{\mathbb{X}}^n \longrightarrow \mathcal{O}_{\mathbb{X}}(1) \longrightarrow \mathcal{O}_W(1) \longrightarrow 0 \tag{E9}$$

where $\mathcal{K}$ is by definition the kernel of the map $\mathcal{O}_{\mathbb{X}}^n \to \mathcal{O}_{\mathbb{X}}(1)$. So, applying π_{1*} to (E9) and assuming that $\mathrm{R}^1\pi_{1*}\big(\mathcal{K}\big) = \mathrm{R}^1\pi_{1*}\big(\mathcal{I}_W(1)\big) = 0$, we have

$$\mathcal{O}_X^n \longrightarrow \pi_{1*}\mathcal{O}_{\mathbb{X}}(1) \longrightarrow \pi_{1*}\mathcal{O}_W(1) \longrightarrow 0.$$

We emphasize that $\mathcal{I}$ is not locally free so $\pi_{1*}(\mathcal{O}_{\mathbb{X}}(1))$ might a priori be different from $\mathcal{I}$ (see Stack project, 26.21. Projective bundles, example 26.21.2). However our strategy is to prove that these coincide in this case.

We use the same notation for the sheaves and their push forward by $\mathbb{X} \overset{\iota}{\hookrightarrow} \mathbb{P}^n_X$. Thus, the strategy is to ensure that $\mathrm{R}^1 p_{1*}(\mathcal{K}) = \mathrm{R}^1 p_{1*}(\mathcal{I}_W(1)) = 0$ and that $p_{1*}(\mathcal{O}_{\mathbb{X}}(1)) = \mathcal{I} \otimes \mathcal{L}$ in order to get the sequence:

$$\mathcal{O}_X^n \longrightarrow \mathcal{I} \otimes \mathcal{L} \longrightarrow p_{1*}\mathcal{O}_W(1) \longrightarrow 0. \tag{E10}$$

As we will explain below, $[p_{1*}\mathcal{O}_W(1)]$ will turn out to be precisely the cycle $[\mathbb{V}({}^{\mathrm{c}}\mathrm{s}_\alpha)]$ which by definition verifies the following exact sequence:

$$\mathcal{E} \xrightarrow{{}^{\mathrm{c}}\mathrm{s}_\alpha} \mathcal{O}_X \longrightarrow \mathcal{O}_{\mathbb{V}({}^{\mathrm{c}}\mathrm{s}_\alpha)} \longrightarrow 0.$$

This will show eventually Theorem 6.1.1.

6.1.2 Cohomological preliminaries

Lemma 6.1.7. *The following vanishings hold:*

(i) $\mathrm{R}^1 p_{1*}\mathcal{I}_{\mathbb{X}}(1) = 0$,

(ii) $\mathrm{R}^{i+1} p_{1*}\mathcal{O}_{\mathbb{X}}(-i) = 0$ *for every* $i \in \{0, \dots, n-1\}$,

(iii) $\mathrm{R}^i p_{1*}\mathcal{O}_{\mathbb{X}}(-i) = 0$ *for every* $i \in \{1, \dots, n-1\}$.

Proof. Under the assumption that $\dim(Z) = 0$, by Corollary 3.2.17, the ideal $\mathcal{I}_{\mathbb{X}}$ has a locally free resolution of the following form:

$$0 \to \mathcal{G}_{n+1} \to \mathcal{G}_n \to \dots \to \mathcal{G}_2 \to \mathcal{G}_1 \to \mathcal{I}_{\mathbb{X}} \to 0 \tag{$\mathcal{G}_\bullet$}$$

where $\mathcal{G}_i = \bigoplus_{j=1}^{i} p^*\mathcal{G}_{ij} \otimes \mathcal{O}_{\mathbb{P}^n_X}(-j)$ when $i \in \{1, \dots, n\}$ and $\mathcal{G}_{n+1} = p^*\mathcal{G}'_{n+1} \otimes \mathcal{O}_{\mathbb{P}^n_X}(-1)$ for some locally free sheaves $\mathcal{G}_{ij}$ and $\mathcal{G}'_{n+1}$ over X.

Now, a diagram chasing in ($\mathcal{G}_\bullet$) shows that $\mathrm{R}^1 p_{1*}\mathcal{I}_{\mathbb{X}}(1) = 0$ provided that $\mathrm{R}^k p_{1*}(\mathcal{G}_k(1)) = 0$ for all $k \in \{1, \dots, n+1\}$. By Proposition 3.1.6, those vanishings are verified since:

- $\mathrm{H}^k(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-j+1)) = 0$ for all $k \in \{1, \dots, n\}$ and all $j \in \{1, \dots, k\}$,
- $\mathrm{H}^{n+1}(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-2)) = 0$

The only non trivial case to check is when $k = n$. But:

$$\mathrm{H}^n(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(-j+1)) \simeq \mathrm{H}^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(j-n-2))^\vee = 0$$

because $j \leq n$.

For *(ii)* and *(iii)*, since $\mathcal{O}_{\mathbb{X}} = \mathcal{O}_{\mathbb{P}^n_X}/\mathcal{I}_{\mathbb{X}}$, the assertions follow from the same argument after twisting the complex ($\mathcal{G}_\bullet$) by $\mathcal{O}_{\mathbb{P}^n_X}(-i)$ for every $i \in \{0, \dots, n-1\}$. □

Lemma 6.1.8. *We have* $p_{1*}(\mathcal{O}_{\mathbb{X}}(1)) = \mathcal{I} \otimes \mathcal{L}$.

Proof. First, $\mathcal{O}_{\mathbb{P}^n_X}(1)$ being the relative ample line bundle of the projective bundle $\mathbb{P}^n_X = \mathbb{P}\big(\mathcal{O}_X^{n+1}\big)$, we have $p_{1*}\mathcal{O}_{\mathbb{P}^n_X}(1) = \mathcal{O}_X^{n+1}$.

Moreover, since $\mathcal{I}_{\mathbb{X}}(1)$ is the image of the canonical map $p_1^*\mathcal{E} \to \mathcal{O}_{\mathbb{P}^n_X}(1)$, we let $\mathcal{H}$ be the kernel of this surjection and we write the exact sequence:

$$0 \to \mathcal{H} \to p_1^*\mathcal{E} \to \mathcal{I}_{\mathbb{X}}(1) \to 0.$$

Since $p_{1*}p_1^*\mathcal{E} \simeq \mathcal{E}$ and $\mathrm{R}^1p_{1*}p_1^*\mathcal{E} = 0$, applying p_{1*} to this exact sequence, we get:

$$0 \to p_{1*}\mathcal{H} \to \mathcal{E} \to p_{1*}\mathcal{I}_{\mathbb{X}}(1) \to \mathrm{R}^1p_{1*}\mathcal{H} \to 0. \tag{a}$$

Also, since we proved that $\mathrm{R}^1p_{1*}\mathcal{I}_{\mathbb{X}}(1) = 0$, applying p_{1*} to the canonical exact sequence

$$0 \to \mathcal{I}_{\mathbb{X}}(1) \to \mathcal{O}_{\mathbb{P}^n_X}(1) \to \mathcal{O}_{\mathbb{X}}(1) \to 0$$

we get

$$0 \to p_{1*}\mathcal{I}_{\mathbb{X}}(1) \to \mathcal{O}_X^{n+1} \to p_{1*}\mathcal{O}_{\mathbb{X}}(1) \to 0. \tag{b}$$

The exact sequences (a) and (b) fit into the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & & \\
 & p_{1*}\mathcal{H} \arrow[d] & 0 \arrow[d] & \\
 & \mathcal{E} \arrow[r, "\simeq"] \arrow[d] & \mathcal{E} \arrow[d] & \\
0 \arrow[r] & p_*\mathcal{I}_{\mathbb{X}}(1) \arrow[r] \arrow[d] & \mathcal{O}_X^{n+1} \arrow[r] \arrow[d] & p_{1*}\mathcal{O}_{\mathbb{X}}(1) \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & \mathrm{R}^1p_{1*}\mathcal{H} \arrow[r] \arrow[d] & \mathcal{I}_Z \otimes \mathcal{L} \arrow[r] \arrow[d] & p_{1*}\mathcal{O}_{\mathbb{X}}(1) \arrow[r] & 0 \\
 & 0 & 0 & &
\end{tikzcd}$$

where (a) is the left column, (b) is the central row and the map $\mathcal{I}_Z \to p_{1*}\mathcal{O}_{\mathbb{X}}(1)$ in the bottom row is the canonical morphism associated to the projectivization of $\mathcal{I}_Z$. This morphism is an isomorphism over $X \backslash Z$ and therefore $\mathcal{I}_Z \otimes \mathcal{L} \to p_{1*}\mathcal{O}_{\mathbb{X}}(1)$ is injective because $\mathcal{I}_Z$ is torsion free. Hence $p_{1*}\mathcal{H} \simeq 0 \simeq \mathrm{R}^1p_{1*}\mathcal{H}$ and $p_{1*}\mathcal{O}_{\mathbb{X}}(1) \simeq \mathcal{I}_Z \otimes \mathcal{L}$.

□

6.1.3 Proof of Theorem 6.1.1

As above, let $W \subset \mathbb{X}$ be the intersection of $\mathbb{X}$ with n general relative hyperplanes of $\mathbb{P}^n_X$ so that $[W] = c_1\big(\mathcal{O}_{\mathbb{X}}(1)\big)^n$.

Proof of Theorem 6.1.1. Consider the following exact sequence:

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{K} \arrow[r] & \mathcal{O}_{\mathbb{X}}^n \arrow[rr, "\beta'"] \arrow[dr] & & \mathcal{O}_{\mathbb{X}}(1) \arrow[r] & \mathcal{O}_W(1) \arrow[r] & 0 \\
& & & \mathcal{I}_W(1) \arrow[ur] \arrow[dr] & & & \\
& & 0 \arrow[ur] & & 0 & &
\end{tikzcd} \tag{Kz}$$

We claim that

$$\mathrm{R}^1 p_{1*}\big(\mathcal{I}_W(1)\big) = \mathrm{R}^1 p_{1*}\big(\mathcal{K}\big) = 0.$$

To prove it, first observe that the Koszul complex

$$0 \longrightarrow \mathcal{O}_{\mathbb{X}}(-n+1) \longrightarrow \ldots \longrightarrow \mathcal{O}_{\mathbb{X}}(-1)^{\binom{n}{2}} \longrightarrow \mathcal{O}_{\mathbb{X}}^n \longrightarrow \mathcal{I}_W(1) \longrightarrow 0$$

is exact. Indeed $\mathbb{X}$ has dimension n since, by Lemma 3.2.8, $\mathbb{X}$ decomposes as the union of $\tilde{X}$, the blow-up of X at $\mathcal{I}$, and the torsion part $\mathbb{T}_Z$, possibly empty, whose reduced structure is $\mathbb{P}^n_{Z'}$ for a set $Z' \subset Z$. Moreover the intersection W in $\mathbb{P}^n_X$ of $\mathbb{X}$ and n generic divisors in $|\mathcal{O}_{\mathbb{P}^n_X}(\xi)|$ (since $\mathcal{O}_{\mathbb{P}^n_X}(\xi)$ is very ample along the fibres) has codimension n in X. Hence the Koszul complex of these sections, restricted to $\mathbb{X}$ is exact.

Then, cutting the Koszul complex into short exact sequence and taking direct images, we see that the desired vanishing holds if we show:

- $\mathrm{R}^{i+1} p_{1*}\mathcal{O}_{\mathbb{X}}(-i) = 0$ for every $i \in \{0, \ldots, n-1\}$,
- $\mathrm{R}^{i} p_{1*}\mathcal{O}_{\mathbb{X}}(-i) = 0$ for every $i \in \{1, \ldots, n-1\}$.

On the other hand, this last vanishing is precisely the content of Lemma 6.1.7 *(ii)* and *(iii)*.

Since $p_{1*}\mathcal{O}_{\mathbb{X}}^n \simeq \mathcal{O}_X^n$, $p_{1*}\mathcal{O}_{\mathbb{X}}(1) \simeq \mathcal{I} \otimes \mathcal{L}$, $\mathrm{R}^1 p_{1*}\big(\mathcal{I}_W(1)\big) = 0$ and $\mathrm{R}^1 p_{1*}(\mathcal{K}) = 0$, pushing forward by p_1 the exact sequence (Kz), we obtain the following commutative diagram:

$$\begin{tikzcd}
& & 0 \arrow[d] & & p_{1*}\mathcal{K} \arrow[d] & & \\
& & \mathcal{O}_X^n \arrow[d, "\beta"] & = & \mathcal{O}_X^n \arrow[d, "p_{1*}\beta'"] & & \\
0 \arrow[r] & \mathcal{E} \arrow[r] \arrow[d, equal] & \mathcal{O}_X^{n+1} \arrow[d, "\alpha"] \arrow[rr, "(\phi_0 \ \ldots \ \phi_n)"] & & \mathcal{I} \otimes \mathcal{L} \arrow[r] \arrow[d] & 0 & \\
& \mathcal{E} \arrow[r, "{}^{c}s_\alpha"] & \mathcal{O}_X \arrow[d] \arrow[rr] & & p_{1*}\mathcal{O}_W(1) \arrow[r] \arrow[d] & 0 & \\
& & 0 & & 0 & &
\end{tikzcd} \tag{D2}$$

where the map α is defined by the diagram (D2) as the cokernel of map β induced from $p_{1*}(\beta')$. The composition $\mathcal{E} \to O_X^{n+1} \xrightarrow{\alpha} \mathcal{O}_X$ gives rise to a cosection s_α and, by the bottom row of (D2), we have $p_{1*}(\mathcal{O}_W) \simeq \mathcal{O}_V(s_\alpha)$. Therefore,

$$[V(s_\alpha)] = p_{1*}[W]. \tag{6.1.2}$$

Now, by definition we have $d_n(\Phi) = \deg(\xi^n|_{\tilde{X}})$. Also, we have

$$\deg(W) = \deg(\xi^n|_{\tilde{X}}) + \deg(\xi^n|_{\mathbb{T}_Z})$$

hence

$$d_n(\Phi) = \deg(W) - \deg(\xi^n|_{\mathbb{T}_Z}).$$

So the statement of the theorem amounts to

$$\deg(W) = \deg\big(\mathbb{V}(s_\alpha)\big).$$

This is guaranteed by (6.1.2).

Since all the general map α as in (E10) can be obtained as cokernel of a general map $\beta : \mathcal{O}_X^n \to \mathcal{O}_X^{n+1}$, $[\mathbb{V}({}^{\mathrm{c}}\mathrm{s}_\alpha)]$ does not depend on the general map α so that we can write $[\mathbb{V}\big({}^{\mathrm{c}}\mathrm{s}(\mathcal{E})\big)]$ for a general cosection ${}^{\mathrm{c}}\mathrm{s}(\mathcal{E})$.

The fact that $\deg(W) = \deg(p_{1*}W)$ comes from the decomposition of W. Indeed, $\mathbb{X}$ decomposes into the graph $\tilde{X}$ and the torsion part $\mathbb{T}_Z$ supported on $\mathbb{P}^n_{\mathrm{Fitt}_{n-1}(Z)}$. Hence, we have the equality

$$[W] = [\mathbb{X}] \cdot c_1\big(\mathcal{O}_{\mathbb{P}^n_X}(1)\big)^n = [\tilde{X}] \cdot c_1\big(\mathcal{O}_{\mathbb{P}^n_X}(1)\big)^n + [\mathbb{T}_Z] \cdot c_1\big(\mathcal{O}_{\mathbb{P}^n_X}(1)\big)^n.$$

Since $\tilde{X}$ is irreducible and $\sigma_1 : \tilde{X} \to X$ birational, we have

$$\deg\big([\tilde{X}] \cdot c_1\big(\mathcal{O}_{\mathbb{P}^n_X}(1)\big)^n\big) = \deg\big(\sigma_{1*}([\tilde{X}] \cdot c_1\big(\mathcal{O}_{\mathbb{P}^n_X}(1)\big)^n)\big).$$

□

Moreover, as a consequence of Theorem 6.1.1 we have:

$$d_t(\Phi) = \deg\big([\mathbb{V}({}^{\mathrm{c}}\mathrm{s}(\mathcal{E})]\big) - \deg\big(p_{1*}([\mathbb{T}_Z] \cdot c_1\big(\mathcal{O}_{\mathbb{P}^n_X}(1)\big)^n)\big).$$

6.2 Study of homaloidal hypersurfaces

6.2.1 Measure of the difference between Rees and symmetric algebras

We relate now the topological degree and the naive topological degree with the notions of Milnor and Tjurina numbers. For the rest of this section, $\mathcal{I}$ is the ideal of a rational map $\Phi = (\phi_0 : \ldots : \phi_n)$ associated to an $n+1$-subspace V of $\mathrm{H}^0(X, \mathcal{L})$ where $\mathcal{L}$ is a line bundle over X. We denote by Z the base scheme $\mathbb{V}(\mathcal{I})$ in X and we assume that $\dim(Z) = 0$.

Generalized Milnor and Tjurina numbers

Notation. We set temporarily as a notation that $\delta^n = \deg\big(c_1(\mathcal{L})^n\big)$ which as to be understood as $\delta = \deg\big(c_1(\mathcal{L})\big)$ when X is the projective space $\mathbb{P}^n$.

Definition 6.2.1. With notation as in Notation 2.2.20, for every $z \in Z$, put:

- $\tau(Z, z) = \mathrm{length}(\mathcal{O}_{Z,z})$
- $\mu(Z, z) = \tau(Z, z) + \deg(T_z)$.

We let $\tau(Z) = \sum_{z\in Z} \tau(Z,z)$ and $\mu(Z) = \sum_{z\in Z} \mu(Z,z)$.

As a direct application of Theorem 6.1.1, we obtain:

Proposition 6.2.2. *The following equalities hold:*

(i) $\deg\left([\mathbb{V}({}^{\mathrm{c}}\mathrm{s}(\mathcal{E}))]\right) = \delta^n - \tau(Z)$

(ii) $d_t(\Phi) = \delta^n - \mu(Z)$

(iii) $d_t(\Phi) - \deg({}^{\mathrm{c}}\mathrm{s}(\mathcal{E})) = \mu(Z) - \tau(Z) = \deg(T) = \deg(p_{1*}T)$.

Proof. Looking back at the diagram (D2), we see that $\mathbb{V}(s_\alpha)$ has the following presentation:

$$\mathcal{O}_X^n \xrightarrow{s_\alpha = (\sum_{i=0}^n a_{i1}\phi_i \ \dots \ \sum_{i=0}^n a_{in}\phi_i)} \mathcal{I}\otimes\mathcal{L} \to \mathcal{O}_{\mathbb{V}(s_\alpha)} \to \ 0$$

where $(a_{ij})_{0\le i\le n, 1\le j\le n}$ is an $(n+1)\times n$ general matrix with entries in the field k. Since by definition, $[\mathbb{V}({}^{\mathrm{c}}\mathrm{s}(\mathcal{E}))] = [\mathbb{V}(s_\alpha)]$ we have that $\deg\left([\mathbb{V}({}^{\mathrm{c}}\mathrm{s}(\mathcal{E}))]\right) = \mathrm{length}(\mathcal{O}_{\mathbb{V}(s_\alpha)}) = \delta^n - \tau(Z)$ by definition of $\tau(Z)$.

The equalities *(ii)* and *(iii)* follow in the same way from the definition of $\mu(Z)$ and $\tau(Z)$ and from the decomposition of $\mathbb{X}$ as the union of $\tilde{X}$ and $\mathbb{T}_Z$. □

The number $\mu(Z)$ being the sum of the numbers $\mu(Z,z)$, we explain how to computationally compute the number $\mu(Z,z)$.

Proposition 6.2.3. *Let $(a_{ij})_{0\le i\le n, 1\le j\le n}$ be an $(n+1)\times n$ general matrix with entries in the field* k. *Then, denoting by $(\sum_{i=0}^n a_{i1}\phi_i, \dots, \sum_{i=0}^n a_{in}\phi_i)_z$ the localisation at z, we have:*

$$\mu(Z,z) = \mathrm{length}(\mathcal{O}_{X,z}/(\sum_{i=0}^n a_{i1}\phi_i, \dots, \sum_{i=0}^n a_{in}\phi_i)_z).$$

Proof. First, since the formation of the symmetric algebra commutes with base change, we can suppose that Z consists of a single point z so that $\mu(Z) = \mu(Z,z)$. Now recall that $d_t(\Phi)$ can be computed in the following way. A general point $y \in \mathbb{P}^n$ is the intersection of n general hyperplanes $L_j : \sum_{i=0}^n a_{ij}x_i = 0$, that is, the data of an $(n+1)\times n$ general matrix N with entry in the field k. The preimage of y by Φ is contained in the scheme $\mathbb{F}' = \mathbb{V}(\sum_{j=0}^n a_{1j}\phi_j, \dots, \sum_{j=0}^n a_{nj}\phi_j)$. Hence, to compute the topological degree of Φ, it remains to remove the points of $\mathbb{F}'$ in the base locus.

So

$$d_t(\Phi) = \mathrm{length}(\mathbb{F}) = \delta^n - \mathrm{length}(\mathcal{O}_{X,z}/(\sum_{i=0}^n a_{i1}\phi_i, \dots, \sum_{i=0}^n a_{in}\phi_i)_z),$$

and since $d_t(\Phi)$ is also equal to $\delta^n - \mu(Z) = \delta^n - \mu(Z, z)$, we have that

$$\text{length}(\mathcal{O}_{X,z}/(\sum_{i=0}^{n} a_{i1}\phi_i, \ldots, \sum_{i=0}^{n} a_{in}\phi_i)_z) = \mu(Z, z).$$

□

As explained to us by Laurent Busé after the defense of this thesis, $\mu_{Z,z}$ can also be interpreted as the multiplicity of the Hilbert-Samuel polynomial of the localization of I at z (see [Eis95, Chapter 12] for the related definitions).

Remark 6.2.4. As it is explained in Example 1.3.9, in practice via Macaulay2, letting

$$\mathbb{F}' = \mathbb{V}(\sum_{j=0}^{n} a_{1j}\phi_j, \ldots, \sum_{j=0}^{n} a_{nj}\phi_j)$$

as in the proof of 6.2.3, the preimage of y is equal to the scheme

$$\mathbb{F} = \mathbb{V}([(\sum_{j=0}^{n} a_{1j}\phi_j, \ldots, \sum_{j=0}^{n} a_{nj}\phi_j) : (\phi_0, \ldots, \phi_n)]^\infty)$$

where, given two ideals J and J' of a ring R, we let $[J : J'^\infty]$ is the saturation of J by J'.

The polar case

In the polar case, X is the projective space $\mathbb{P}^n$ over k.

Definition 6.2.5. Let $F = \{f = 0\}$ be a hypersurface in $\mathbb{P}^n$ where f is a homogeneous polynomial of degree d in $\mathrm{k}[x_0, \cdots, x_n]$. Let $f_i = \frac{\partial f}{\partial x_i}$ and $\mathcal{I} = (f_0, \ldots, f_n)$ be the ideal sheaf in $\mathcal{O}_{\mathbb{P}^n}$ generated by the partial derivatives of f, called the *jacobian ideal* of f. Recall that we call the map Φ_f associated to $\mathcal{I}$ the *polar map*.

The topological degree of Φ_f is called the *polar degree of* F.

In order to use the Euler identity $d \cdot f = x_0 f_0 + \ldots + x_n f_n$, we suppose in the sequel that the characteristic of the base field does not divide the degree of the polynomial f defining the hypersurface F. We also always assume that the jacobian ideal $\mathcal{I}$ of F is zero-dimensional.

We recall the classical definition of Milnor and Tjurina numbers.

Definition 6.2.6. Let $z \in Z = \mathbb{V}(\mathcal{I})$ and via a change of coordinates, suppose that $z = (1 : 0 : \ldots : 0)$. Set $g_\flat \in \mathrm{k}[x_1, \ldots, x_n]$, the usual deshomogeneisation of a homogeneous polynomial $g \in \mathrm{k}[x_0, \cdots, x_n]$ in the chart $\{x_0 \neq 0\}$.

The *local Tjurina number* at z, denoted by $\tau_f(Z, z)$ is defined as

$$\tau_f(Z, z) = \text{length}\big(\mathcal{O}_{\mathrm{k}^n,z}/(f_\flat, (f_\flat)_1, \ldots, (f_\flat)_n)\big) \quad \text{where } (f_\flat)_i = \frac{\partial f_\flat}{\partial x_i}.$$

The *local Milnor number* at z, denoted by $\mu_f(Z, z)$, is defined as

$$\mu_f(Z, z) = \text{length}\big(\mathcal{O}_{\mathrm{k}^n,z}/((f_\flat)_1, \ldots, (f_\flat)_n)\big) \quad \text{where } (f_\flat)_i = \frac{\partial f_\flat}{\partial x_i}.$$

The *global Tjurina number* of F, denoted by $\tau_f(Z)$ (resp. *global Milnor number* of F, denoted by $\mu_f(Z)$) is the sum $\sum \tau_f(Z,z)$ (resp. $\sum \mu_f(Z,z)$) over all $z \in Z$.

We explain now how the numbers $\mu(Z)$ and $\tau(Z)$ defined in Definition 6.2.1 coincide with the usual definitions of Milnor and Tjurina number given in Definition 6.2.6.

Proposition 6.2.7. *Let $F = \{f = 0\}$ be a reduced hypersurface in $\mathbb{P}^n$ where f is a homogeneous polynomial in $\mathrm{k}[x_0, \cdots, x_n]$ of degree d. Let $z \in Z = \mathbb{V}(\mathcal{I})$ then:*

$$\tau(Z,z) = \tau_f(Z,z) \quad \text{and} \quad \mu(Z,z) = \mu_f(Z,z).$$

Proof. Via a change of coordinates, we can suppose $z = (1 : 0 : \ldots : 0)$. The deshomogenisation of the Euler identity in the chart $\{x_0 \neq 0\}$ is:

$$d \cdot f_\flat = (f_0)_\flat + \sum_{i=1}^{n} x_i (f_i)_\flat$$

and $(f_i)_\flat = (f_\flat)_i$ for $1 \leq i \leq n$. The equality

$$((f_0)_\flat, \ldots, (f_n)_\flat) = (f_\flat, (f_\flat)_1, \ldots, (f_\flat)_n)$$

implies that $\tau(Z,z) = \tau_f(Z,z)$.

For the Milnor number, we let $A = (a_{ij})_{0 \leq i \leq n, 1 \leq j \leq n}$ a general $(n+1) \times n$ matrix with entries in the field k. By Proposition 6.2.3,

$$\mu(Z,z) = \mathrm{length}\big(\mathcal{O}_{\mathbb{P}^n,z} / (\sum_{i=0}^{n} a_{i1} f_i, \ldots, \sum_{i=0}^{n} a_{in} f_i)_z\big).$$

By localisation at z, we have that $\mu(Z,z) = \mathrm{length}(\mathcal{O}_{M_A})$ where M_A is defined as the cokernel of the following composition map:

$$\mathcal{O}_z^n \xrightarrow{A} \mathcal{O}_z^{n+1} \xrightarrow{(f_0 \ \ldots \ f_n)_z} \mathcal{O}_z \longrightarrow \mathcal{O}_{M_A} \longrightarrow 0$$

whereas $\mu_f(Z,z) = \mathrm{length}(\mathcal{O}_M)$ where M is defined as the cokernel of the following composition map:

$$\mathcal{O}_z^n \xrightarrow{\begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & & \vdots \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}} \mathcal{O}_z^{n+1} \xrightarrow{(f_0 \ \ldots \ f_n)_z} \mathcal{O}_z \longrightarrow \mathcal{O}_M \longrightarrow 0.$$

But, since $\mathrm{rank}(A) = n$, we have $\mathrm{length}(\mathcal{O}_{M_A}) = \mathrm{length}(\mathcal{O}_M)$.

□

In the case when $\tau(Z,z) = \mu(Z,z)$ for a point $z \in Z$, Z is also called quasi-homogeneous at z in [Sai80]. As an application of the previous proposition, we recover a result originally proved over the field $\mathbb{C}$ in [DP03].

Proposition 6.2.8. *Let $F = \{f = 0\} \subset \mathbb{P}^n$ be a reduced hypersurface of degree d over an algebraically closed field* k*. Let $\Phi_f = (f_0 : \ldots : f_n)$ be the polar map of f and assume that $\mathbb{V}(f_0, \ldots, f_n)$ is finite.*

Then

$$d_t(\Phi_f) = (d-1)^n - \mu_f(Z).$$

Proof. Since f has degree d and $\mathbb{V}(f_0, \ldots, f_n)$ is finite, Proposition 6.2.8 follows from Proposition 6.2.7 and Proposition 6.2.2 *(ii)* since the polynomials f_i have degree $d-1$. □

6.2.2 Proofs of Proposition 19 and Proposition 20

In characteristic 3, a homaloidal curve of degree 5

We give eventually the proof of Proposition 19. Recall the result.

Proposition 6.2.9. *The curve $F = \mathbb{V}\big((x_1^2+x_0x_2)x_0(x_1^2+x_0x_2+x_0^2)\big)$ is homaloidal if and only if the base field* k *has characteristic* 3*, in which case the inverse of the polar map is*

$$\Psi = (x_1^2x_2^2 + x_0x_2^3 + x_2^4 : -x_1^3x_2 - x_0x_1x_2^2 - x_1x_2^3 : -x_1^4 - x_0x_1^2x_2 + x_0x_2^3)$$

Proof. Here all the computations of resolutions, descriptions of the projectivization of the ideal sheaf, of the graph of the polar, of the torsion components or of the Milnor/Tjurina numbers were made using Macaulay2. The curve F is defined over $\mathbb{Z}$ hence over $\mathbb{F}_p$ for every p. Given the characteristic p of k, we will denote the polar of F by Φ_p.

First, let us explain why the polar of F cannot be birational if $p > 20$. The idea here is that the polar has the same behaviour in high enough characteristic than in characteristic 0. In order to do so we follow the presentation in [Ngu16]. When $p > 20$, remark that the base locus of Φ_p is support over $z = (0:0:1)$ because the reduction of the equation of F modulo p does not affect its coefficients.

Now, recall that the conductor invariant $\delta(Z, z)$ is defined as the length of the quotient module $\overline{\mathcal{O}_{F,z}}/\mathcal{O}_{F,z}$, where $\overline{\mathcal{O}_{F,z}}$ is the normalization of the local ring $\mathcal{O}_{F,z}$. Let $r(F, z)$ denote the number of local branches of F at z. Using étale cohomology, Deligne showed (cf. [Del73], [MHW01]) that

$$\mu(Z, z) = 2\delta(Z, z) - r(F, z) + 1 + \mathrm{Sw}(F, z)$$

where $\mathrm{Sw}(F, z)$ denotes the Swan conductor of F at z (see [Del73, 1.7, 1.8] for the definition).

In our case, we have after computation that $\delta(Z, z) = 8$ and $r(F, z) = 3$ and by [Ngu16, Corollary 3.2], we have that $\mathrm{Sw}(F) = 0$ if $p > \kappa(F)$ where $\kappa(F) = \dim\big(\mathcal{O}_{\mathbb{P}^2,z}\big)/(f_\flat, \alpha f_{x_0} + \beta f_{x_1})$ where f_{x_0} and f_{x_1} are the partial derivatives of the deshomogeneisation $f_\flat$ of $(x_1^2 + x_0x_2)x_0(x_1^2 + x_0x_2 + x_0^2)$ with respect of x_2 and $(\alpha : \beta) \in \mathbb{P}^1$ is generic. Hence $\kappa(F) \leq 20$.

So, if $p > 20$, we have that $\mu(Z, z) = 14$ so $d_2(\Phi_p) = 2$ and F is not homaloidal.

Concerning the remaining case $2 \leq p \leq 20$, we proceed as follows. The resolution of the jacobian ideal $\mathcal{I}$ over $\mathbb{Q}$ is as follows:

$$0 \to \mathcal{O}(-1) \oplus \mathcal{O}(-3) \xrightarrow{\begin{pmatrix} 0 & 2x_0^3+4x_0x_1^2+4x_0^2x_2 \\ x_0 & -x_1^3 \\ -2x_1 & -6x_0x_1^2-8x_0^2x_2-8x_1^2x_2-6x_0x_2^2 \end{pmatrix}} \mathcal{O}^3 \to \mathcal{I}(4) \to 0 \tag{R11}$$

where we denote $\mathcal{O}$ for the sheaf $\mathcal{O}_{\mathbb{P}^2}$.

By a case by case computation, we observe that for every prime $p \neq 2$ and $p \neq 5$ the reduction modulo p of (R11) provides a resolution of $\mathcal{I}_p = \mathcal{I} \otimes_{\mathbb{Z}} \mathbb{F}_p$. In every characteristic $20 \geq p \geq 3$ different from 5, $\mathrm{Fitt}_1 \mathcal{I}_p = (x_0, x_1)$ so $\mathcal{I}_p$ is not a complete intersection and $\mathbb{P}(\mathcal{I}_p)$ has a torsion component above the point $z = (0:0:1) \in \mathbb{P}^2$.

Moreover, in characteristic other than 2 and 5, the resolution of $\mathbb{X}_p = \mathbb{P}(\mathcal{I}_p)$ embedded in $\mathbb{P}^n \times \mathbb{P}^n$ is as follow:

$$0 \longrightarrow \mathcal{O}(-4,-2) \longrightarrow \mathcal{O}(-1,-1) \oplus \mathcal{O}(-3,-1) \longrightarrow \mathcal{I}_{\mathbb{X}_p} \longrightarrow 0$$

where $\mathcal{O}$ stands for $\mathcal{O}_{\mathbb{P}^n \times \mathbb{P}^n}$ and we wrote to the right the shift in the variables of the second factor of the product $\mathbb{P}^n \times \mathbb{P}^n$. From this resolution, we can compute that $\tau(Z, z) = 13$ in every characteristic other that 2 and 5.

In characteristic 3, $\mathcal{I}_3$ has the following resolution:

$$0 \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-3) \xrightarrow{\begin{pmatrix} 0 & x_0^3 - x_0x_1^2 - x_0^2x_2 \\ x_0 & x_1^3 \\ x_1 & -x_0^2x_2 - x_1^2x_2 \end{pmatrix}} \mathcal{O}^3 \longrightarrow \mathcal{I}_3(4) \longrightarrow 0.$$

The difference in characteristic 3 comes from the multiplicity of the torsion component in $\mathbb{X}_3$. Indeed, the torsion component $\mathbb{T}_{Z_p}$ has the following resolution over $\mathbb{F}_p$ for $5 < p \leq 20$.

$$0 \longrightarrow \mathcal{O}(-2,0) \longrightarrow \mathcal{O}(-1,0)^2 \longrightarrow \mathcal{I}_{\mathbb{T}_{Z_p}} \longrightarrow 0$$

whereas in characteristic 3, it has resolution:

$$0 \longrightarrow \mathcal{O}(-3,-1) \longrightarrow \begin{matrix} \mathcal{O}(-3,0)^2 \\ \oplus \\ \mathcal{O}(-2,-1)^2 \end{matrix} \longrightarrow \begin{matrix} \mathcal{O}(-2,0)^3 \\ \oplus \\ \mathcal{O}(-1,-1) \end{matrix} \longrightarrow \mathcal{I}_{\mathbb{T}_{Z_3}} \longrightarrow 0.$$

Given the generators of $\mathcal{I}_{\mathbb{T}_{Z_3}}$ (respectively $\mathcal{I}_{\mathbb{T}_{Z_p}}$ in characteristic $5 < p \leq 20$), we compute that the multidegree of $\mathbb{T}_{Z_3}$ (resp. $\mathbb{T}_{Z_p}$) is $(2,0,0)$ (resp. $(1,0,0)$). Hence the difference $\mu(Z) - \tau(Z)$ is equal to 2 in characteristic 3 (resp. $\mu(Z) - \tau(Z) = 1$ if $p > 5$) so the Milnor number $\mu(Z, z)$ is equal to 15 and $d_2(\Phi_3) = 1$ in characteristic 3 or else $\mu(Z, z) = 14$ and $d_2(\Phi_p) = 2$ for $p > 5$. In characteristic 3, the polar map can be written

$$\Phi_f = (x_1^4 + x_0^3x_2 + x_0x_1^2x_2 : -x_0^3x_1 + x_0x_1^3 + x_0^2x_1x_2, x_0^4 - x_0^2x_1^2 - x_0^3x_2).$$

and it is a computation to check that Ψ is the inverse of Φ_f. As we will see in Remark 7.2.6, the inverse Ψ of Φ is actually the polar map of the dual curve F.

When $p = 2$ or 5, the polar of F is not dominant. □

Remark 6.2.10. What we did is to deepen the multiplicity of the torsion component by specializing the resolution of $\mathcal{I}$ over $\mathbb{Z}$ modulo a prime p for which some monomials of the presentation matrix disappear (here $p = 3$ works). More precisely, let O_t be the subscheme of $\mathbb{P}^2$ defined by $(x_0, x_1)^t$. Hence $M_{|O_1} = 0$ in any characteristic. But if $\mathrm{char}(\mathrm{k}) \neq 3$, $M_{|O_2}$ has two non zero columns, one of linear entries and one of quadratic entries whereas if $\mathrm{char}(\mathrm{k}) = 3$ the quadratic column-s vanishes. Hence $\mathbb{P}(\mathcal{I}_Z(4)_{|O_2})$ is different with the characteristic of k. In other words, in characteristic 3, the torsion part $\mathbb{T}_Z$ is not equal scheme-theoretically to $\mathbb{P}^n_{\mathrm{Fitt}_{n-1}\mathcal{I}}$ contrary to what happens in greater characteristic. It is not clear if such an example is sporadic or not which leads to the following questions.

Problem F. *Given a field* k *of positive characteristic, are the homaloidal curves of* $\mathbb{P}^2_{\mathrm{k}}$ *of bounded degree? What is the classification of such curves?*

Reduction problem in positive characteristic

Proposition 6.2.11. *Let k be an algebraically closed field of characteristic* 101.

(i) The curve $\mathbb{V}\big(z(y^3 + x^2z)\big)$ *has polar degree* 2 *whereas* $\mathbb{V}\big(z^{50}(y^3 + x^2z)^{51}\big)$ *has polar degree* 1.

(ii) The curve $\mathbb{V}\big((y^3+x^2z)(y^2+xz)\big)$ *has polar degree* 5 *whereas the curve* $\mathbb{V}\big((y^3+x^2z)^{31}(y^2 + xz)^4\big)$ *has polar degree* 3.

The analysis of the presentation of the jacobian ideal gives also an easy way to construct examples of non reduced plane curves in positive characteristic where the topological degree is not preserved by reduction. It suffices to compute the presentation matrix of the jacobian ideal and adjust the characteristic of the field in order to modify the first syzygy matrix.

Proof. All the computations of resolution were made using Macaulay2.

Both curves are defined over $\mathbb{Z}$ and as in the proof of Proposition 6.2.9, the idea is to take reduction modulo the prime $p = 101$ of the resolution of their jacobian ideal over $\mathbb{Z}$ to get a resolution over $\mathbb{F}_p$. We give the complete argument for Item *(i)*. Item *(ii)* is similar and left to the reader. As in the proof of Proposition 6.2.9, $\mathcal{I}_p$ stands for $\mathcal{I} \otimes_{\mathbb{Z}} \mathbb{F}_p$.

The jacobian ideal of $\mathbb{V}\big(z(y^3 + x^2z)\big) = 0$ has resolution

$$0 \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}^3 \xrightarrow{\Phi_{red}} \mathcal{I}_{101}(3) \longrightarrow 0,$$

$\mathcal{I}_{\mathbb{X}_{101}}$ has the following resolution:

$$0 \longrightarrow \mathcal{O}(-3,-2) \longrightarrow \begin{matrix} \mathcal{O}(-1,-1) \\ \oplus \\ \mathcal{O}(-2,-1) \end{matrix} \longrightarrow \mathcal{I}_{\mathbb{X}_{101}} \longrightarrow 0.$$

There is no torsion component above the point $z = (1 : 0 : 0)$ and so the corresponding polar map has topological degree 2.

But the jacobian ideal of the curve $\mathbb{V}\big(z^{50}(y^3 + x^2 z)^{51}\big)$ has resolution

$$0 \longrightarrow \mathcal{O}(-1) \oplus \mathcal{O}(-2) \longrightarrow \mathcal{O}^3 \xrightarrow{\Phi} \mathcal{I}_{101}(3) \longrightarrow 0$$

and $\mathcal{I}_{\mathbb{X}_{101}}$ has the following resolution:

$$0 \longrightarrow \mathcal{O}(-3,-2) \longrightarrow \begin{array}{c} \mathcal{O}(-1,-1) \\ \oplus \\ \mathcal{O}(-2,-1) \end{array} \longrightarrow \mathcal{I}_{\mathbb{X}_{101}} \longrightarrow 0.$$

There is a torsion component above the point $z = (1 : 0 : 0)$, what we can see from the resolution of $\tilde{X}$:

$$0 \longrightarrow \mathcal{O}(-2,-2)^2 \longrightarrow \begin{array}{c} \mathcal{O}(-1,-1) \\ \oplus \\ \mathcal{O}(-2,-1) \\ \oplus \\ \mathcal{O}(-1,-2) \end{array} \longrightarrow \mathcal{I}_{\tilde{X}} \longrightarrow 0.$$

The polar map of the latter curve is given by

$$(x : y : z) \mapsto (xz^2 : -49y^2 z : 50y^3)$$

and its inverse is $(x : y : z) \mapsto (-37xz^2 : -3y^2 z : y^3)$. $\square$

Chapter 7

Inverse of a birational map

Suppose we are given a birational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$. Then how should we compute the inverse Φ^{-1}? The goal of this chapter is to give an answer to this question, provided we have certain data on the presentation of the ideal of the base locus Z. To illustrate the method we propose, we focus on finding the inverse of the following birational map in characteristic 3 which is the polar map of $f = (x_1^2 + x_0x_2)x_0(x_1^2 + x_0x_2 + x_0^2)$.

$$\begin{aligned} \Phi_{ex} : \quad & \mathbb{P}^2 \dashrightarrow \mathbb{P}^2 \\ & (x_0 : x_1 : x_2) \dashrightarrow (x_1^4 + x_0^3x_2 + x_0x_1^2x_2 : -x_0^3x_1 + x_0x_1^3 + x_0^2x_1x_2 : \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad x_0^4 - x_0^2x_1^2 - x_0^3x_2) \end{aligned}$$

Recall that if $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ is birational, the inverse Φ^{-1} is also given by $n+1$ homogeneous polynomials $\phi'_0, \ldots, \phi'_n \in \mathrm{k}[x_0, \cdots, x_n]$ of the same degree δ' such that

$$\big(\phi_0(\phi'_0, \ldots, \phi'_n), \ldots, \phi_n(\phi'_0, \ldots, \phi'_n)\big) = (x_0P, \ldots, x_nP)$$

for a homogeneous polynomial $P \in \mathrm{k}[x_0, \cdots, x_n]$ and

$$\big(\phi'_0(\phi_0, \ldots, \phi_n), \ldots, \phi'_n(\phi_0, \ldots, \phi_n)\big) = (x_0Q, \ldots, x_nQ)$$

for a homogeneous polynomial $Q \in \mathrm{k}[x_0, \cdots, x_n]$ what we commonly denote by $\Phi \circ \Phi^{-1} = \Phi^{-1} \circ \Phi = id$, see Section 1.3.

We emphasize that if the polynomials $\phi_i = \sum_{j=0}^{n} a_{ij}x_j$ are of degree 1 for all $i \in \{0, \ldots, n\}$, then Φ is completely defined by the matrix $A = (a_{ij})_{0 \le i,j \le n}$ and the problems concerning the birational properties of Φ or concerning the inverse of Φ are equivalent to solve the following linear system

$$(E) \left\{ \begin{array}{ccccc} y_0 & = & a_{00}x_0 + & \cdots & + a_{0n}x_n \\ \vdots & & \vdots & \vdots & \vdots \\ y_n & = & a_{n0}x_0 + & \cdots & + a_{nn}x_n \end{array} \right.$$

that is, Φ is birational if and only if A is invertible and in this case Φ^{-1} is the map associated to the matrix A^{-1}. So the problem of finding the inverse of birational map can be taken as a generalisation of a classical problem in linear algebra.

After presenting in Section 7.1 the currently available techniques to compute Φ^{-1}, we will point out a method based on our analysis of torsion components. The idea is that, while the inverse is essentially known once we have the bigraded generators of the ideal of the Rees algebra (corresponding to the blow-up Γ of X along Z), the computation may fail because these generators are sometimes inaccessible. On the other hand, the symmetric algebra (corresponding to $\mathbb{X} = \mathbb{P}(\mathcal{I}_Z)$) is much easier to deal with and essentially it is computed from the presentation of $\mathcal{I}_Z$. Also, we know that Γ is obtained from $\mathbb{X}$ by removing the torsion components, which are supported over the schemes defined by the Fitting ideals of $\mathcal{I}_Z$. Then, it suffices to saturate the ideal of $\mathbb{X}$ with respect to the Fitting ideals of $\mathcal{I}_Z$ (pulled back to $\mathbb{P}^n \times \mathbb{P}^n$) to obtain the desired generators.

For the rest of the chapter, the settings are as follows.

Notation 7.0.1. Let $\Phi : \mathbb{P}^n_1 \dashrightarrow \mathbb{P}^n_2$ be a dominant map given by $n+1$ homogeneous polynomials $\phi_0, \ldots, \phi_n \in \mathrm{k}[x_0, \cdots, x_n]$ of the same degree δ and without common factor. Let also $R_1 = \mathrm{k}[x_0, \ldots, x_n]$ (resp. $R_2 = \mathrm{k}[y_0, \ldots, y_n]$) be the coordinate ring of $\mathbb{P}^n_1$ (resp. $\mathbb{P}^n_2$) and denote by I_Z the ideal of R_1 of the base locus Z of Φ.

Denote also by $S = R_1 \otimes R_2$ the coordinate ring of $\mathbb{P}^n_1 \times \mathbb{P}^n_2$. We identify S with the space of homogeneous polynomials in both variables $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$ and an element $P \in S$ of degree d_1 in the $\mathbf{x}$ variables of degree d_2 in the $\mathbf{y}$ variables is said of *bidegree* (d_1, d_2).

In the case that Φ is birational, we let $\Phi^{-1} : \mathbb{P}^n_2 \dashrightarrow \mathbb{P}^n_1$ given by $\phi'_0, \ldots, \phi'_n \in \mathrm{k}[y_0, \ldots, y_n]$ be the inverse of Φ with base locus ideal $I_{Z'}$.

7.1 State of the art to compute the inverse

Let us adapt first the notions and notation of Section 2.1 and Section 2.2 in the context of Notation 7.0.1. Let M be the presentation matrix of I_Z and denote by $\bigoplus_{i\geq 1} R_1^{n_i}(-d_i) \to R_1^{n+1}$ the graded map defined by M. Our convention is that we do not include pieces of the form R_1^0 in the direct sum $\bigoplus_{i\geq 1} R_1^{n_i}(-d_i)$.

The ideal $I_{\mathbb{X}}$ of $\mathbb{X} = \mathbb{P}(I_Z)$ in $\mathbb{P}^n_1 \times \mathbb{P}^n_2$ is thus generated by polynomials of bidegree $(d_i, 1)$ for i such that $n_i \neq 0$. More precisely, there is a surjection

$$\bigoplus_{i\geq 1} S^{n_i}(-d_i, -1) \to I_{\mathbb{X}} \to 0.$$

Example 7.1.1. Let ϕ_0, ϕ_1, ϕ_2 be the three polynomials generating the base ideal I_Z of Φ_{ex}. A minimal free resolution of I_Z reads

$$0 \to R_1(-1) \oplus R_1(-3) \xrightarrow{M = \begin{pmatrix} 0 & x_0^3 - x_0x_1^2 - x_0^2x_2 \\ x_0 & x_1^3 \\ x_1 & -x_0^2x_2 - x_1^2x_2 \end{pmatrix}} R_1^{n+1} \to I_Z(4) \to 0$$

and $I_{\mathbb{X}}$ verifies:

$$S(-1,-1) \oplus S(-3,-1) \to I_{\mathbb{P}(I_Z)} \to 0.$$

Moreover, as we saw in Section 2.1, the graph Γ_Φ of Φ is isomorphic to the blow-up of $\mathbb{P}_1^n$ along Z such that we have the following commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}_1^n \times \mathbb{P}_2^n \arrow[d, hook] \arrow[ddl, "p_1"'] \arrow[ddr, "p_2"] & \\
 & \mathbb{P}(I_Z) \arrow[d, hook] \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & \\
 & \Gamma \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\
\mathbb{P}_1^n \arrow[rr, dashed, "\Phi"'] & & \mathbb{P}_2^n
\end{tikzcd}$$

where p_1 (resp. p_2) is the projection over the first (resp. second) factor.

Now assume that Φ is birational of inverse Φ^{-1}. Of course $\Gamma = \mathrm{Graph}(\Phi) = \mathrm{Graph}(\Phi^{-1})$ but $\mathbb{X}' = \mathbb{P}(\mathcal{I}_{Z'})$ might differ from $\mathbb{X}$. This is clear from the study of a given set of generator of the ideal $I_{\mathbb{X}'}$ in $\mathbb{P}_1^n \times \mathbb{P}_2^n$. Indeed, as for Φ, let M' be the presentation matrix of $I_{Z'}$ and denote by $\bigoplus_{i\geq 1} R_2^{n_i'}(-d_i') \to R_2^{n+1}$ the graded map defined by M' and there is a surjection

$$\bigoplus_{i\geq 1} S^{n_i}(-1,-d_i') \to I_{\mathbb{P}(I_{Z'})} \to 0.$$

Hence provided that M (or M') does not have only linear entries, $\mathbb{P}(I_Z)$ and $\mathbb{P}(I_{Z'})$ differ. In this case, they both differ also from Γ since the ideal I_Γ of Γ in $\mathbb{P}_1^n \times \mathbb{P}_2^n$ contains stricly $I_{\mathbb{P}(I_Z)}$ and $I_{\mathbb{P}(I_{Z'})}$.

We summarize the situation into the following commutative diagram

$$\begin{tikzcd}
 & \mathbb{P}_1^n \times \mathbb{P}_2^n \arrow[dl, bend right, "p_1"'] \arrow[dr, bend left, "p_2"] & \\
 & \mathbb{X} \arrow[u, hook] \arrow[d, "\pi_1"'] & \mathbb{X}' \arrow[u, hook] \arrow[d, "\pi_2'"] \\
 & \Gamma \arrow[ul, hook] \arrow[ur, hook] \arrow[dl, "\sigma_1"'] \arrow[dr, "\sigma_2"] & \\
\mathbb{P}_1^n \arrow[rr, dashed, "\Phi"] & & \mathbb{P}_2^n \arrow[ll, dashed, bend left, "\Phi^{-1}"]
\end{tikzcd}$$

where, for the clarity of the diagram, we did not mention the map $\pi_2 : \mathbb{X} \to \mathbb{P}_2^n$ (resp. $\pi_1' : \mathbb{X}' \to \mathbb{P}_1^n$) which is the restriction of the second projection p_2 to $\mathbb{X}$ (resp. of the first projection p_1 to $\mathbb{X}'$).

7.1.1 Inverse of the standard Cremona map

Recall that the *standard Cremona map* τ is defined as follows

$$\begin{array}{rcl} \tau : & \mathbb{P}^n_1 & \dashrightarrow \mathbb{P}^n_2 \\ & (x_0 : \ldots : x_n) & \dashrightarrow (\phi_0 : \ldots : \phi_n) \end{array}$$

where $\phi_i = x_0 \ldots x_{i-1} x_{i+1} \ldots x_n$ for $i \in \{0, \ldots, n\}$.

It is a computation to show that the standard Cremona map is an involution i.e. that $\tau \circ \tau = id$. Our goal here is to explain a general method to recover this result.

The ideal $I_Z = (\phi_0, \ldots, \phi_n)$ of the base locus Z of τ has the following minimal free presentation:

$$M = \begin{pmatrix} x_0 & 0 & \cdots & 0 \\ -x_1 & x_1 & \cdots & \vdots \\ 0 & -x_2 & \cdots & \vdots \\ \vdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & x_{n-1} \\ 0 & 0 & \cdots & -x_n \end{pmatrix}$$

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^n}(-1)^n \xrightarrow{\qquad\qquad} \mathcal{O}^{n+1}_{\mathbb{P}^n} \longrightarrow \mathcal{I}_Z(n) \longrightarrow 0$$

Hence, denoting $y_0, \ldots, y_n$ the variables of $\mathbb{P}^n_2$, the ideal sheaf $\mathcal{I}_{\mathbb{X}}$ of the projectivization $\mathbb{X}$ of $\mathcal{I}_Z$ in $\mathbb{P}^n_1 \times \mathbb{P}^n_2$ is generated by the entries in the row matrix $\begin{pmatrix} y_0 & \ldots & y_n \end{pmatrix} M$ i.e.

$$I_{\mathbb{X}} = (y_0 x_0 - y_1 x_1, \ldots, y_{n-1} x_{n-1} - y_n x_n).$$

The remark is that, in this precise case, $I_{\mathbb{X}}$ is also generated by the entries of the row matrix:

$$\begin{pmatrix} x_0 & \ldots & x_n \end{pmatrix} \begin{pmatrix} y_0 & 0 & \cdots & 0 \\ -y_1 & y_1 & \cdots & \vdots \\ 0 & -y_2 & \cdots & \vdots \\ \vdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & y_{n-1} \\ 0 & 0 & \cdots & -y_n \end{pmatrix}$$

and we can reverse the construction of the projectivization $\mathbb{X}$ from the ideal $\mathcal{I}_Z$. Indeed, since the inverse τ^{-1} of τ has also a base locus Z' in $\mathbb{P}^n_2$ with ideal $I_{Z'}$, we can embed the projectivization $\mathbb{X}'$ of $I_{Z'}$ into $\mathbb{P}^n_1 \times \mathbb{P}^n_2$ via the presentation matrix M' of $I_{Z'}$. The ideal $I_{\mathbb{X}'}$ is generated by the entries in the row matrix $\begin{pmatrix} x_0 & \ldots & x_n \end{pmatrix} M'$. So here a candidate to be the presentation matrix of $I_{Z'}$ is the matrix:

$$M' = \begin{pmatrix} y_0 & 0 & \cdots & 0 \\ -y_1 & y_1 & \cdots & \vdots \\ 0 & -y_2 & \cdots & \vdots \\ \vdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & y_{n-1} \\ 0 & 0 & \cdots & -y_n \end{pmatrix}$$

and indeed, in the case of the standard map, taking the $n \times n$-minors of M', we recover the map τ', up to a change of coordinates i.e. we recover the base ideal $I_{Z'}$ of τ', the inverse of τ.

To recover completely τ', it remains to identify this last change of coordinates. In order to do it, one solution is to compute $n+1$ $n \times n$-minors of M' denoted by $\phi''_0, \ldots, \phi''_n$ (we denote τ'' the map associated to these generators of $I_{Z'}$), and to write down explicitly the condition on a change of coordinates associated to a matrix $A = (a_{ij})_{0 \leq i,j \leq n} \in \mathrm{PGl}_{n+1}(\mathrm{k})$. That is equivalent to compute the image $y^0 = \tau(x^0), \ldots y^{n+1} = \tau(x^{n+1})$ of $n+2$ general points $x^0, \ldots x^{n+1}$ by τ, to compute the image $x^{0\prime\prime} = \tau''(y^0), \ldots, x^{n+1\prime\prime} = \tau''(y^{n+1})$ and to compute the automorphism of $\mathbb{P}^n$ sending the projective basis $x^{0\prime\prime}, \ldots, x^{n+1\prime\prime}$ to the projective basis $x^0, \ldots, x^{n+1}$ (this last step, way simpler that what we originally thought of was explained to us by Laurent Manivel).

7.1.2 Jacobian dual

Of course the previous example is very special. For instance, it is of linear type and the ideal of the blow-up Γ (or of $\mathbb{X}$) is only generated in bi-degree $(1,1)$ i.e. is generated by linear forms with respect to the $\mathbf{x}$ and $\mathbf{y}$ variables. But we give here a sketchy presentation of a result due to F.Russo and A.Simis, extending to more general situations this previous method [RS01]. For the sake of concision and to keep easily the analogy with the standard map, we only present the result in the case of a dominant rational map $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ but we emphasize that it extends in greater generality.

Let $\Phi : \mathbb{P}^n_1 \dashrightarrow \mathbb{P}^n_2$ be a dominant rational map given by $n+1$ homogeneous polynomials $\phi_0, \ldots, \phi_n \in \mathrm{k}[x_0, \cdots, x_n]$ of degree δ and let

$$R_1^m \xrightarrow{M} R_1^{n+1} \longrightarrow I_Z(\delta) \longrightarrow 0$$

be a free presentation of the ideal I_Z of the base locus Z of Φ in $\mathbb{P}^n$. We decompose the presentation matrix M as the concatenation of a matrix M_1 whose columns are the columns of M with linear entries and a matrix M_2 made by the other columns of M. We emphasize that, denoting $y_0, \ldots, y_n$ the variables of $\mathbb{P}^n_2$, the entries of the row matrix $\begin{pmatrix} y_0 & \ldots & y_n \end{pmatrix} M_1$ are the equations (of bi-degree $(1,1)$) of the projectivization $\mathbb{X}$ of I_Z in $\mathbb{P}^n_1 \times \mathbb{P}^n_2$.

Following [RS01], we also denote by Θ the Jacobian matrix of the row matrix $\begin{pmatrix} y_0 & \ldots & y_n \end{pmatrix} M_1$ with respect to the variables $x_0, \ldots, x_n$ of $\mathbb{P}^n_1$ and q stands for the number of columns of M_1. Let $\mathcal{Z}(\Theta)$ be the kernel of $\Theta : R_2^{n+1} \to R_2^q$. Given

a vector $\mathbf{g} = (g_0, \dots, g_n)$, consider the morphism $R_1 \to R_2$ which sends x_i to g_i for $i \in \{0, \dots, n-1\}$ and apply it to the entries of the matrix M_1. The result is a matrix with entries in R_2 which we will denote by $M_1(\mathbf{g})$.

Definition 7.1.2. The ideal I_Z is said to have the *strong rank property* if the matrix Θ has rank at most n and, for some minimal homogeneous generator $\mathbf{g} \in \mathcal{Z}(\Theta)$, the evaluated matrix $M_1(\mathbf{g})$ has rank n.

Remark 7.1.3. In Definition 7.1.2, it is implicit that the *strong rank property* of an ideal does not depend on the choice of the vector $\mathbf{g}$. The result is even stronger [RS01, Proposition 1.2, Supplement]. It states that any two such vectors $\mathbf{g}$ and $\mathbf{g}'$ are proportional.

Looking back to the case of the standard Cremona map, we have that $M_1 = M$, $M_2 = 0$,

$$\Theta = \begin{pmatrix} y_0 & 0 & \cdots & 0 \\ -y_1 & y_1 & \cdots & \vdots \\ 0 & -y_2 & \cdots & \vdots \\ \vdots & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & y_{n-1} \\ 0 & 0 & \cdots & -y_n \end{pmatrix}$$

and given $\mathbf{g} = (y_1 \dots y_n, \dots, y_0 \dots y_{n-1})$ (which is in $\mathcal{Z}(\Theta)$), we have that $M_1(\mathbf{g})$ has rank $n = \operatorname{rank}(M)$. Hence I_Z has the *strong rank property* and the result is that Φ is birational with an inverse given by the vector $\mathbf{g}$. This is the content of the following proposition which is a combination of [RS01, Theorem 1.4] and [RS01, Proposition 2.1].

Proposition 7.1.4. *Let $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$ be a dominant rational map, M be the presentation matrix of the ideal I_Z of the base locus Z of Φ and denote as above by M_1 the linear part of M.*

Then Φ is birational and $\operatorname{rank}(M) = \operatorname{rank}(M_1)$ *if and only if I_Z has the strong rank property.*

Moreover, the inverse of Φ is given by any vector $\mathbf{g} \in \mathcal{Z}(\Theta)$.

As far as we know, this result is the first to give a computationally very efficient method to construct the inverse of a birational map and this method was indeed the first method implemented in the Macaulay2 package "Cremona" in order to invert birational maps [Sta17]. However, there exists birational maps Φ whose base locus ideals I_Z do not have the strong rank property, for instance Φ_{ex}. Hence we present now an alternative computational method to invert birational maps.

7.2 Computation of the inverse

We refer to Notation 7.0.1 for all the notation we use and we assume that the map $\Phi : \mathbb{P}^n_1 \dashrightarrow \mathbb{P}^n_2$ is birational.

Example 7.2.1. Over a field of characteristic different from 2, the map Φ_{ex} is birational but its base locus ideal I_Z does not have the strong rank property. Indeed, a presentation of I_Z reads

$$0 \longrightarrow R_1(-1) \oplus R_1(-3) \xrightarrow{\begin{pmatrix} 0 & x_0^3 - x_0x_1^2 - x_0^2x_2 \\ x_0 & x_1^3 \\ x_1 & -x_0^2x_2 - x_1^2x_2 \end{pmatrix}} R_1^3 \longrightarrow I_Z(4) \longrightarrow 0.$$

and the matrix M_1 of columns with entries of degree 1 of the presentation matrix M of I_Z has rank 1 which is smaller than 2.

Proposition 7.2.2. *The polynomials of bidegree* $(1,1)$ *generating* $I_{\mathbb{P}(I_Z)}$ *in* $\mathbb{P}_1^n \times \mathbb{P}_2^n$ *are in one-to-one correspondence with the polynomials of bidegree* $(1,1)$ *generating* $I_{\mathbb{P}(I_{Z'})}$.

Proof. Let $\begin{pmatrix} l_0 \\ \vdots \\ l_n \end{pmatrix} = \begin{pmatrix} a_{00}x_0 + \ldots + a_{0n}x_n \\ \vdots \\ a_{n0}x_0 + \ldots + a_{nn}x_n \end{pmatrix}$ be a linear syzygy of I_Z seen as a module over R_1 generated by $(\phi_0, \ldots, \phi_n)$.

Hence, since $I_{\mathbb{P}(I_Z)} \subset I_{\mathrm{Graph}(\Phi)}$, the polynomial $y_0(a_{00}x_0 + \ldots + a_{0n}x_n) + \ldots + y_n(a_{n0}x_0 + \ldots + a_{nn}x_n)$ is a generator of $I_{\mathrm{Graph}(\Phi)}$ over $S = R_1 \otimes R_2$. Since

$$\begin{aligned} y_0(a_{00}x_0 + \ldots + a_{0n}x_n) + \ldots + y_n(a_{n0}x_0 + \ldots + a_{nn}x_n) = \\ x_0(a_{00}y_0 + \ldots a_{n0}y_n) + \ldots + x_n(a_{0n}y_0 + \ldots + a_{nn}y_n) \end{aligned}$$

and that $\mathrm{Graph}(\Phi) = \mathrm{Graph}(\Phi^{-1})$, we have that $\begin{pmatrix} a_{00}y_0 + \ldots a_{n0}y_n \\ \vdots \\ a_{0n}y_0 + \ldots + a_{nn}y_n \end{pmatrix}$ is a (linear) syzygy of the base locus ideal $I_{Z'}$ of Φ^{-1}.

Hence, given a minimal presentation of I_Z i.e. the presentation matrix M of I_Z does not have entry of degree 0, all the columns of degree 1 provide independent columns of degree 1 in the presentation matrix M' of $I_{Z'}$ and there is no other column of degree 1 in M' otherwise by the reversibility of the construction, it would provide other columns of degree 1 in M. □

If we denote by $I_{(1,1)}$ the ideal of polynomials of bidegree $(1,1)$ generating $I_{\mathbb{P}(I_Z)}$ (or $I_{\mathbb{P}(I_{Z'})}$), the subscheme $L = \mathbb{V}(I_{(1,1)})$ of $\mathbb{P}_1^n \times \mathbb{P}_2^n$ contains both $\mathbb{P}(I_Z)$ and $\mathbb{P}(I_{Z'})$ and when I_Z has the strong rank property, L and $\mathbb{P}(I_Z)$ have the same dimension even if they are not equal.

In the case that I_Z has the strong rank property, the result is that we can compute the inverse of Φ without computing the equations of $\mathrm{Graph}(\Phi)$ i.e. without computing the Rees algebra $\mathcal{R}(I_Z)$ of $\mathcal{I}_Z$. In this perspective, the algorithm we propose for the case that I_Z do not have the strong rank property is a method of computing the Rees algebra $\mathcal{R}(I_Z)$. We present the algorithm before commenting it.

Algorithm. (1) *Compute the presentation matrix* M *of* I_Z *over* R_1.

$$\bigoplus_{i\geq 0} R_1(-i)^{a_i} \xrightarrow{M} R_1^{n+1} \longrightarrow I_Z(\delta) \longrightarrow 0$$

(2) *Over $S = R_1 \otimes R_2$, form the ideal $I_{\mathbb{P}(I_Z)}$ generated by the entries in the row matrix $\begin{pmatrix} y_0 & \dots & y_n \end{pmatrix} M$.*

(3) *Compute the saturation $\bigoplus_{i=1}^{n-1} [I_{\mathbb{P}(I_Z)} : p_1^* I_i(M)^\infty]$ of $I_{\mathbb{P}(I_Z)}$. Since the ideal $I_i(M)$ are the Fitting ideals whose support are the torsion components, the final ideal is the ideal I_Γ of the blow-up of I_Z. Another way to compute I_Γ is to compute the primary decomposition of $I_{\mathbb{P}(I_Z)}$ and to choose the component of the blow-up Γ. This latter method reveals to be computationally expensive.*

(4) *Form the ideal $I_{\mathbb{P}(I_{Z'})}$ generated by the generators of bidegree $(1, d')$ of I_Γ and denote by N the row matrix of those generators. The ideal $I_{\mathbb{P}(I_{Z'})}$ is the ideal of $\mathbb{P}(I_{Z'})$.*

$$\bigoplus_{i\geq 0} S(-1,-i)^{b_i} \xrightarrow{N} I_{\mathbb{P}(I_{Z'})} \longrightarrow 0$$

(5) *Let M' be the jacobian matrix of N with respect to the variables $x_0, \dots, x_n$. The matrix M' is the presentation matrix $I_{Z'}$.*

(6) *Compute the generators of the cokernel. If M' is a $(n+1) \times n$-matrix, this is the same as computing the $n \times n$-minors of M'. If M' is bigger, this is the same as computing the syzygies matrix of the transpose ${}^tM'$ of M'.*

(7) *Compute the change of coordinates in order to find generators $\phi'_0, \dots, \phi'_n$ of $I_{Z'}$ such that $\Phi^{-1} = (\phi'_0, \dots, \phi'_n)$ as we explained in the study case of the standard Cremona map.*

Before commenting the algorithm, we right down its implementation on the computer system Macaulay2 with the map Φ_{ex}.

Example 7.2.3. We give the detailed interaction with Macaulay2. Let us emphasize that we present the computation over a base field of characteristic 3 because it was originally our motivation and the last step of computation of the polynomials $\phi'_0, \dots, \phi'_n$ is way more easier in this case.

(0) We start by the definitions of the field, rings, ideal. Here I stands for the ideal I_Z.

```
i1 : k = ZZ/3;

i2 : R1 = k[x_0..x_2];

i3 : R2 = k[y_0..y_2];

i4 : I = ideal (x_1^4+x_0^3*x_2+x_0*x_1^2*x_2,-x_0^3*x_1+
x_0*x_1^3+x_0^2*x_1*x_2, x_0^4-x_0^2*x_1^2-x_0^3*x_2);
```

(1) We compute the presentation matrix M of I_Z.

```
o4 : Ideal of R1

i5 : M = syz gens I

o5 = {4} | 0   x_0^3-x_0x_1^2-x_0^2x_2 |
     {4} | x_0 x_1^3                   |
     {4} | x_1 -x_0^2x_2-x_1^2x_2      |

               3        2
o5 : Matrix R1  <--- R1
```

(2) We compute now the primary decomposition of $I_{\mathbb{P}(I_Z)}$. In the code, we denote by J the ideal $I_{\mathbb{P}(I_Z)}$ and by IG the ideal I_Γ.

```
i6 : S = R1**R2; --tensor product of R1 and R2

i7 : J = ideal (matrix{{y_0,y_1,y_2}}*sub(M,S) );

o7 : Ideal of S

i8 : time primaryDecomposition J
     -- used 0.110233 seconds

                                 2      3                 2
o8 = {ideal (x y  + x y , x y y  + x y  + x y y y  - x y y
              0 1    1 2   0 0 2    1 1    1 0 1 2    2 1 2
     -----------------------------------------------------
       2     3    2        2 2    2                      2
- x y y - x y  , x y y  - x y  - x y y  - x x y y  - x x y
   2 0 2   2 2    0 0 2    1 1    1 0 2    0 2 0 2    0 2 2
-----------------------------------------------------------------
             3        2       3      2        2        2
+ x x y y , x y  - x x y  +  x y  - x x y  - x x y  - x x y ),
   1 2 1 2   0 0    0 1 0     1 1    0 2 0    0 2 2    1 2 2
-----------------------------------
                      2         2
 ideal (x y  + x y , x , x x , x )}
         0 1    1 2   1   0 1   0

o8 : List

i9 : IG = (primaryDecomposition J)_0

                                 2      3                 2
o9 = ideal (x y  + x y , x y y  + x y  + x y y y  - x y y
             0 1    1 2   0 0 2    1 1    1 0 1 2    2 1 2
```

```
     ----------------------------------------------------
       2      3   2        2 2    2                      2
- x y y  - x y , x y y  - x y  - x y y  - x x y y  - x x y
   2 0 2    2 2   0 0 2    1 1    1 0 2    0 2 0 2    0 2 2
-----------------------------------------------------------
             3        2      3      2        2        2
+ x x y y , x y  - x x y  + x y  - x x y  - x x y  - x x y )
   1 2 1 2   0 0    0 1 0    1 1    0 2 0    0 2 2    1 2 2

o9 : Ideal of S
```

(4) Now, the ideal J′ stands for $I_{\mathbb{P}(I_{Z'})}$.

```
i10 :
      J' = ideal ( (gens IG)_(0,0) , (gens IG)_(0,1) );

o10 : Ideal of S
```

(5) We directly right down the matrix M'.

```
i11 : M' = matrix{{y_1,y_0*y_2^2},{y_2,y_1^3+y_0*y_1*y_2},
{0,-y_1^2*y_2-y_0*y_2^2-y_2^3}};

              3       2
o11 : Matrix S  <--- S
```

and we consider M′ as a matrix of R_2.

(6) Even if the ideal $I_{Z'}$ is the ideal of the 2×2-minors of M', we compute the generators of $I_{Z'}$ with the following more general method:

```
i12 : syz transpose M'

o12 = | -y_1^2y_2^2-y_0y_2^3-y_2^4    |
      | y_1^3y_2+y_0y_1y_2^2+y_1y_2^3 |
      | y_1^4+y_0y_1^2y_2-y_0y_2^3    |

              3       1
o12 : Matrix S  <--- S
```

where, as for M′, we consider the latter matrix with entries in R_2. Its entries are the generators of $I_{Z'}$.

(7) For the final step consisting in finding the polynomials $\phi'_0, \ldots, \phi'_n$, let us not follow the strategy of finding the change of coordinates by posing the conditions (this strategy amounts to compute the inverse of a 12×12 matrix after choosing 4 general points of $\mathbb{P}^2$). The fact is that, in characteristic 3, we can find this change of coordinates by testing first some values. And indeed, starting from

$$\Phi'' = (-y_1^2y_2^2 - y_0y_2^3 - y_2^4 : y_1^3y_2 + y_0y_1y_2^2 + y_1y_2^3 : y_1^4 + y_0y_1^2y_2 - y_0y_2^3)$$

we find that the inverse of Φ is

$$\Phi_{ex}^{-1} = (y_1^2y_2^2 + y_0y_2^3 + y_2^4 : -y_1^3y_2 - y_0y_1y_2^2 - y_1y_2^3 : -y_1^4 - y_0y_1^2y_2 + y_0y_2^3).$$

Remark 7.2.4. If the ultimate goal is to find explicitly the polynomials $\phi'_0 \dots, \phi'_n$ of Φ_{ex}^{-1}, we emphasize that the two methods we detailed, namely to pose the conditions over the change of coordinates or to be tricky, require a lot of time. However, as we will see, Item (7) of the algorithm is fastly handled in some MACAULAY2 packages (though it is unclear to us what methods are followed by these packages).

Remark 7.2.5. As we explained in Item (3) of the algorithm, computing the primary decomposition of $I_{\mathbb{P}(I_Z)}$ is equivalent to express the scheme $\mathbb{P}(I_Z)$ as the union of Γ and of other torsion components. But, in our cases of rational maps $\Phi : \mathbb{P}^n \dashrightarrow \mathbb{P}^n$, we know the shape of those torsion components since they are supported over the base scheme Z of Φ. In the case of Φ_{ex} the base scheme is supported over the point $\mathbb{V}(x_0, x_1)$, hence $\mathbb{P}(I_Z)$ is the union of Γ with a torsion part $\mathbb{T}_Z$ whose reduced structure is isomorphic to $\mathbb{P}^2_{(0:0:1)}$. Hence the ideal of Γ is the saturation of the ideal of $\mathbb{P}(I_Z)$ by the ideal $I_{\mathbb{T}_Z} = (x_0, x_1)$. The gain is that the computation of the saturation in is less expansive than the computation of the primary decomposition in Item (2).

```
i13 :
      time saturate(J, ideal(x_0,x_1) )
     -- used 0.00715082 seconds

                                  2      3                  2
o13 = ideal (x y  + x y , x y y  + x y  + x y y y  - x y y
              0 1    1 2   0 0 2    1 1    1 0 1 2    2 1 2
            ------------------------------------------------
        2      3   2        2 2    2                      2
 - x y y  - x y , x y y  - x y  - x y y  - x x y y  - x x y
    2 0 2    2 2   0 0 2    1 1    1 0 2    0 2 0 2    0 2 2
-----------------------------------------------------------------
              3        2      3      2        2        2
 + x x y y , x y  - x x y  + x y  - x x y  - x x y  - x x y )
    1 2 1 2   0 0    0 1 0    1 1    0 2 0    0 2 2    1 2 2

o13 : Ideal of S
```

Let us mentions that several MACAULAY2 packages, for instance "Cremona" and "RationalMaps" packages provide the possibility to compute the inverse of any birational map and not only those with a base ideal having the strong rank property. However, it is not clear from our study of those packages if they use the blow-up algebra and if so, how do they compute it.

For us, a problem prolonging the concept of the strong rank property is the following. As we saw in Proposition 7.2.2, the generators of bidegree $(1, 1)$ of the ideal of $\mathbb{P}(I_Z)$ are the generators of bidegree $(1, 1)$ of the ideal of $\mathbb{P}(I_{Z'})$. Hence, it seems to us that the ideal of $\mathbb{P}(I_{Z'})$ should be easier to compute in the case

that $I_{\mathbb{P}(I_Z)}$ is generated by some polynomials of bidegree $(1,1)$ or equivalently, if I_Z has some linear syzygies. In this direction, the *strong rank property* would be the optimal case in which the ideal $\mathbb{P}(I_{Z'})$ is the most easily computable.

We finish this chapter with the following remark.

Remark 7.2.6. Recall that, as we explained at the beginning of the chapter, in characteristic 3, the map Φ_{ex} is the polar map of the curve $F = \mathbb{V}\big((x_1^2+x_0x_2)(x_1^2+x_0x_2+x_0^2)x_0\big)$. Consider now the dual curve F' of F. It is actually the union of the dual curve C_1' of $C_1 = \mathbb{V}(x_1^2+x_0x_2)$, C_2' of $\mathbb{V}(x_1^2+x_0x_2+x_0^2)$ and $V(y_2)$ the dual of the singular point of F. Here we write y_0, y_1, y_2 for the variables of the dual $\mathbb{P}^{2\prime}$. To compute C_1', we compute the hessian of $x_1^2+x_0x_2$ which is equal in characteristic 3 to $H = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$. Hence $C_1' = \mathbb{V}\big(\begin{pmatrix} y_0 & y_1 & y_2 \end{pmatrix} H^t \begin{pmatrix} y_0 & y_1 & y_2 \end{pmatrix}\big) = \mathbb{V}(-y_1^2 - y_0y_2)$ and in the same way $C_2' = \mathbb{V}(-y_1^2 - y_0y_2 + y_2^2)$.

Considering the polar $\Phi_{ex'}$ of F', we remark that $\Phi_{ex'} = \Phi_{ex}^{-1}$. The fact that the inverse of the polar map is the polar of the dual curve is actually true for all the homaloidal plane curve we know at the moment (i.e. the three homaloidal curves in any characteristic different from 2, and the one in characteristic 3).

It could be interesting to investigate if this phenomenon is more general.

Bibliography

[BCJ09] L. Busé, M. Chardin, and J.-P. Jouanolou. Torsion of the symmetric algebra and implicitization. *Proc. Amer. Math. Soc.*, 137:1855–1865, 2009.

[BCS10] L. Busé, M. Chardin, and A. Simis. Elimination and nonlinear equations of the rees algebra. *J. Algebra*, 324:1314–1333, 2010.

[BH93] W. Bruns and J. Herzog. *Cohen-Macaualy rings*. Cambridge Studies in Adv. Math., 1993.

[Bou70] N. Bourbaki. *Algèbre, Livre 1 3*. Élements de Mathématique. Springer-Verlag Berlin Heidelberg, 2007 (1970).

[BV88] W. Bruns and U. Vetter. *Determinantal Rings*. Lecture Notes in Mathematics 1327. Springer-Verlag Berlin Heidelberg, 1988.

[CLO07] D.A. Cox, J. Little, and D. O'Shea. *Ideals, Varieties, and Algorithms*. Undergraduate Texts in Mathematics. Springer-Verlag New York, 2007.

[CRS08] C. Ciliberto, F. Russo, and A. Simis. Homaloidal hypersurfaces and hypersurfaces with vanishing hessian. *Adv. Math.*, 218:1759–1805, 2008.

[Del73] P. Deligne. La formule de milnor. *SGA 7 II*, Sém. Géom. Algébrique du Bois-Marie 1967-1969, Lecture Notes in Math. 340, Expose XVI:197–211, 1973.

[Dés12] J. Déserti. *Some properties of the Cremona group*, volume 21 of *Ens. Matemáticos [Mathematical Surveys]*. Sociedade Brasileira de Matemática, Rio de Janeiro, 2012.

[DH17] J. Déserti and F. Han. Quarto-quartic birational maps of $\mathbb{P}^3(\mathbb{C})$. *Internat. J. Math.*, 28, 2017.

[DHS12] A.V. Dória, S.H. Hassanzadeh, and A. Simis. A characteristic free criterion of birationality. *Adv. Math.*, 230:390–413, 2012.

[Dim15] A. Dimca. Freeness versus maximal global tjurina number for plane curves. *Preprint available at arXiv:1508.04954*, [math.AG], 2015.

[Dol00] I.V. Dolgachev. Polar Cremona transformation. *Michigan Math. J.*, 48:191–202, 2000.

[Dol11] I.V. Dolgachev. *Classical Algebraic Geometry: a modern view.* Cambridge Univ. Press, 2011.

[DP03] A. Dimca and S. Papadima. Hypersurface complements, milnor fibers and higher homotopy groups of arrangements. *Ann. Math.*, 158:473–507, 2003.

[dPW99] A.A. du Plessis and C.T.C. Wall. Application of the theory of the discriminant to higly singular plane curves. *Math. Proc. Camb. Phil. Soc.*, 126:256–266, 1999.

[DS15] A. Dimca and G. Sticlaru. Nearly free divisors and rational cuspidal curves. *Preprint available at arXiv:1505.00666v3*, [math.AG], 2015.

[EH00] D. Eisenbud and J. Harris. *The Geometry of Schemes.* Graduate Texts in Mathematics. Springer-Verlag New York, 2000.

[Ein93] L. Ein. On the cohomology of projectively CohenMacaulay determinantal subvarieties of P^n. *in Geometry of complex projective varieties (Cetraro, 1990) (A. Lanteri, M. Palleschi, D. C. Struppa, eds.)*, pages 143–152, Sem. Conf., 9, Mediterranean, Rende, 1993.

[Eis95] D. Eisenbud. *Commutative algebra, with a view toward algebraic geometry.* Graduate Texts in Mathematics. Springer, 1995.

[Eis05] D. Eisenbud. *The Geometry of Syzygies.* Graduate Texts in Mathematics. Springer-Verlag New York, 2005.

[FM12] T. Fassarella and N. Medeiros. On the polar degree of projective hypersurfaces. *J London Math. Soc.*, 86 (1):259–271, 2012.

[Har77] R. Hartshorne. *Algebraic Geometry.* Graduate Texts in Mathematics. Springer, 1977.

[Har80] R. Hartshorne. Stable reflexive sheaf. *Mathematische Annalen*, 254(2):121–176, 1980.

[Har92] J. Harris. *Algebraic Geometry.* Graduate Texts in Mathematics. Springer, 1992.

[Huh14] J. Huh. Milnor numbers of projective hypersurfaces with isolated singularities. *Duke Math. J.*, 163:1525–1548, 2014.

[Man13] L. Manivel. Prehomogeneous spaces and projective geometry. *Rend. Sem. Mat. Univ. Politec. Torino*, 71:35–118, 2013.

[MHW01] A. Melle-Hernández and C.T.C. Wall. Pencils of curves on smooth surfaces. *Proc. Lond. Math. Soc.*, III. Ser. 83 no. 2:257278, 2001.

[Mic64] A. Micali. Sur les algèbre universelles. *Ann. Inst. Fourier*, 14:33–87, 1964.

[Mil68] J. Milnor. *Singular points of complex hypersurfaces*, volume 61 of *Ann. of Math. Stud.* Princeton Univ. Press, 1968.

[MV17] S. Marchesi and J. Vallès. Nearly free curves and arrangements: a vector bundle point of view. *arXiv:1712.04867 [math.AG]*, 2017.

[Ngu16] H.D. Nguyen. Invariants of plane curve singularities and Plücker formulas in positive characteristic. *Ann. Inst. Fourier*, 66(5):2047–2066, 2016.

[Pan99] I. Pan. Une remarque sur la génération du groupe de Cremona. *Bol. Soc. Brasil. Mat. (N.S.)*, 30(1):95–98, 1999.

[PR13] L. Pirio and F. Russo. Quadro-quadric cremona maps and varieties 3-connected by cubics: semi-simple part and radical. *Internat. J. of Math.*, 24:fascicolo 13, 2013.

[PR14] L. Pirio and F. Russo. Quadro-quadric cremona transformations in low dimensions via the jc-correspondence. *Ann. de l' Inst. Fourier*, 64:71–111, 2014.

[PR16] L. Pirio and F. Russo. The xjc-correspondence. *J. fuer die reine und angewandte Math. (Crelle's Journal)*, 716:229–250, 2016.

[RS01] F. Russo and A. Simis. On birational maps and jacobian matrices. *Compositio Math.*, 126:335358, 2001.

[Sai80] K. Saito. Theory of logarithmic differential forms and logarithmic vector fields. *J. Fac. Sci. Univ. Tokyo*, 27:268–291, 1980.

[Sch03] H. Schenck. *Computational algebraic geometry.* London Mathematical Society student texts 58. Cambridge University Press, 2003.

[SJ68] J.G. Semple and Tyrrell J.A. Specialization of cremona transformation. *Mathematika 15*, pages 171–177, 1968.

[SST18] D. Siersma, J. Steenbrink, and Tibăr. Polar degree and Huh's conjectures. *Arxiv 1805.08175*, 2018.

[Sta17] G. Stagliano. A Macaulay2 package for computations with rational maps. *Preprint available at arXiv:1203.5417*, [math.AG], 2017.

[Vas05] W. Vasconcelos. *Integral closure, Rees Algebras, Multiplicities, Algorithms.* Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[Wey03] J. Weyman. *Cohomology of vector bundles and syzygies.* Cambridge Tracts in Mathematics no. 149. Cambridge University Press, 2003.

Résumé

Dans cette thèse, nous interprétons géométriquement la torsion de l'algèbre symétrique d'un faisceau d'idéaux $\mathcal{I}_Z$ d'un schéma Z défini par $n+1$ équations dans une variété n-dimensionnelle. Ceci revient à étudier la géométrie de la projectivisation de $\mathcal{I}_Z$. Les applications de ce point de vue concernent en particulier le domaine des transformations birationnelles de l'espace projectif de dimension 3 au sujet duquel nous construisons des transformations birationnelles explicites qui ont le même degré algébrique que leur inverse, le domaine des courbes libres et presque-libres au sujet duquel nous généralisons une caractérisation des courbes libres en étendant les notions de nombre de Milnor et de nombre de Tjurina. Nous abordons aussi le sujet des hypersurfaces homaloides, notre motivation initiale, au sujet duquel nous exhibons en particulier une courbe homaloide de degré 5 en caractéristique 3. La dernière application concerne le calcul de l'inverse d'une transformation birationnelle.

Mots clés: Géométrie algébrique, Algèbre commutative, Théorie des singularités, Transformations birationelles, Hypersurfaces homaloïdes, courbes libres et presque libres, algèbre de Rees et algèbre symmétrique, Syzygies, Résolutions

Title of the thesis : Geometry of the projectivization of ideals and applications to problems of birationality

Abstract

In this thesis, we interpret geometrically the torsion of the symmetric algebra of the ideal sheaf $\mathcal{I}_Z$ of a scheme Z defined by $n+1$ equations in an n-dimensional variety. This is equivalent to study the geometry of the projectivization of $\mathcal{I}_Z$. The applications of this point of view concern, in particular, the topic of birational maps of the projective space of dimension 3 for which we construct explicit birational maps that have the same algebraic degree as their inverse, free and nearly-free curves for which we generalise a characterization of free curves by extending the notion of Milnor and Tjurina numbers. We tackle also the topic of homaloidal hypersurfaces, our original motivation, for which we produce in particular a homaloidal curve of degree 5 in characteristic 3. The last application concerns the computation of the inverse of a birational map.

Keywords: Algebraic Geometry, Commutative algebra, Singularity theory, Birational maps, Homaloidal hypersurfaces, free and nearly free curves, Symmetric and Rees algebra, Syzygies, Resolutions